AN INTRODUCTION TO MINIMAL SURFACES

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Abstract. The following note provides an introduction to the theory of minimal surfaces. We motivate relevant concepts using graph surfaces in $\mathbb{R}^3$, and then generalize them to Riemannian manifolds. We next investigate the first and second variations of the volume functional, and eventually present several implications of the formulas we obtained. Most of the materials are drawn from [2] and [5], but many details are reconstructed.

Contents

1. Motivation of the Theory

For the beginning of the theory, let us focus on the specific setting of hypersurfaces in $\mathbb{R}^3$. A surface M in $\mathbb{R}^3$ is minimal if for every p in M, there is a neighborhood around p bounded by a simple closed curve such that the surface area of this neighborhood is the smallest among all surfaces bounded by this curve. Heuristically speaking, a minimal surface is a surface that locally minimizes surface area.

We now perform some computations to give a quantitative insight into minimal surfaces. Suppose $u : \Omega \to \mathbb{R}$ is a C^2 function over some domain $\Omega \subset \mathbb{R}^2$, and we consider the graph of the function u given by

$$\mathrm{Graph}_u = \{(x, y, u(x, y)) : (x, y) \in \Omega\} \tag{1.1}$$

Then the area of the graph is given by

$$\begin{aligned}\mathrm{Area}(\mathrm{Graph}_u) &= \int_\Omega |(1, 0, u_x) \times (0, 1, u_y)| \, dA \\ &= \int_\Omega \sqrt{1 + u_x^2 + u_y^2} \, dA \\ &= \int_\Omega \sqrt{1 + |\nabla u|^2} \, dA\end{aligned} \tag{1.2}$$

and we note that the upward pointing normal is given by

$$N = \frac{(-u_x, -u_y, 1)}{\sqrt{1+|\nabla u|^2}} \tag{1.3}$$

Now, we consider a one-parameter family of graphs $u + t\eta$. Here $\eta|_{\partial\Omega} = 0$ to ensure that $u|_{\partial\Omega} = u + t\eta|_{\partial\Omega}$. Then

$$\text{Area}(\text{Graph}_{u+t\eta}) = \int_\Omega \sqrt{1 + |\nabla u + t\nabla\eta|^2}\ dA \tag{1.4}$$

Direct computation gives

$$\begin{aligned} \frac{d}{dt}\bigg|_{t=0} \text{Area}(\text{Graph}_{u+t\eta}) &= \int_\Omega \frac{\langle \nabla u, \nabla \eta\rangle}{\sqrt{1+|\nabla u|^2}}\ dA \\ &= -\int_\Omega \eta\ \text{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right)\ dA \end{aligned} \tag{1.5}$$

where the second equality above follows from integration by parts. Therefore, the graph of u is a critical point of the area functional if and only if u satisfies the divergence form **minimal surface equation (MSE)**

$$\text{div}\left(\frac{\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0 \tag{1.6}$$

Expanding the left-hand side we also get

$$0 = (1 + u_y^2)u_{xx} + (1 + u_x^2)u_{yy} - 2u_x u_y u_{xy} \tag{1.7}$$

Remark 1.8. When $|\nabla u|$ is bounded, MSE is a uniformly elliptic equation, so we can use some classical theory to study its properties.

Now, we need to show that solutions to (MSE) indeed minimize area.

Lemma 1.9. *If $u : \Omega \to \mathbb{R}$ satisfies the minimal surface equation and $\Sigma \subset \Omega \times \mathbb{R}$ is any other surface with $\partial\Sigma = \partial(\text{Graph}_u)$, then*

$$\text{Area}(\text{Graph}_u) \leqslant \text{Area}(\Sigma)$$

Proof. Let N be the normal vector to Graph_u as in (3.1), and we may extend the domain of N to $\Omega \times \mathbb{R}$ by setting $N(x, y, z) = N(x, y)$. By the fact that N is constant with respect to z-variable and (MSE),

$$\text{div}_{\mathbb{R}^3}(N) = \text{div}\left(\frac{-\nabla u}{\sqrt{1+|\nabla u|^2}}\right) = 0$$

Note that the divergence theorem gives us

$$\int_{\text{Graph}_u} N \cdot N_{\text{Graph}_u} - \int_\Sigma N \cdot N_\Sigma = \int_{\text{Vol btw }\Omega\text{ and }\Sigma} \text{div}_{\mathbb{R}^3}(N) = 0$$

Then

$$\text{Area}(\text{Graph}_u) = \int_{\text{Graph}_u} N \cdot N_{\text{Graph}_u} = \int_\Sigma N \cdot N_\Sigma \leqslant \text{Area}(\Sigma)$$

where the last inequality is because $N \cdot N_\Sigma \leqslant 1$. □

At this point we have seen that solutions to (MSE) give rise to minimal surfaces.

2. Geometry of Submanifolds

In this section, we review some important concepts in Riemannian geometry so that we can generalize concepts related to minimal surfaces to minimal submanifolds, which we shall define in the next section. Let $\Sigma \subset M$ be a submanifold. For every vector field X on Σ, we use $X^\top$ and $X^\perp$ to denote the tangential and normal components of X, respectively. We then define the connection on Σ by

$$(\nabla_\Sigma)_X Y = (\nabla_X Y)^\top$$

and the second second fundamental form on Σ by

$$A(X, Y) = (\nabla_X Y)^\perp$$

We note that the second fundamental form is a symmetric bilinear form, since the Lie bracket of two vector fields tangential to Σ is again tangential to Σ.

Let $x \in \Sigma$, and $\{\partial_i\}_i$ an orthonormal frame near x. We define the norm squared of the second fundamental form by

$$|A|^2 = \sum_{i,j=1}^{k} |A(\partial_i, \partial_j)|^2$$

and we also use the notation

$$A_{ij} = |A(\partial_i, \partial_j)|$$

The mean curvature vector H at x is given by

$$H = \sum_{i=1}^{k} A(\partial_i, \partial_i)$$

and the divergence of X at $x \in \Sigma$ is given by

$$\operatorname{div}_\Sigma(X) = \sum_{i=1}^{n} \langle \nabla_{\partial_i} X, \partial_i \rangle$$

It could be checked that the definition of mean curvature is independent of the orthonormal frame chosen. We note that

$$\operatorname{div}_\Sigma(Y^\perp) = \sum_i \langle \partial_i, \nabla_{\partial_i} Y^\perp \rangle = -\sum_i \langle Y^\perp, \nabla_{\partial_i} \partial_i \rangle = -\langle Y^\perp, H \rangle$$

so the definition of divergence is also independent of the orthonormal frame chosen.

Eventually, given a smooth function f on Σ, we define the gradient $\nabla_\Sigma f$ at $x \in \Sigma$ by

$$\langle \nabla_\Sigma f, X \rangle_x = df_x(X)$$

for every vector field X on Σ. Then the Laplace operator Δ_Σ on Σ is given by

$$\Delta_\Sigma f = \operatorname{div}_\Sigma(\nabla_\Sigma f)$$

As in Eulidean space $\mathbb{R}^n$, we say that f is harmonic at x if

$$\Delta_\Sigma f = 0$$

at x.

3. First and Second Variations

Let $F : \Sigma \times (-\varepsilon, \varepsilon) \to M^n$ be a variation of Σ with compact support and fixed boundary. That is, $F = \mathrm{Id}$ outside of a compact set, $F(x, 0) = x$ for all $x \in \Sigma$, and $F(x, t) = x$ for all $x \in \partial\Sigma$. The vector field F_t restricted to Σ is often called the variational vector field.

Now we want to compute the first variation of area for this one-parameter family of surfaces. Let $\{x_i\}_i$ be local coordinates on Σ, and we denote by ∂_i the local frame associated with $\{x_i\}_i$. Since $F(\cdot, t)$ is diffeomorphic onto its image for every $t \in (-\varepsilon, \varepsilon)$, we may pushforward the frame to the image $F(\Sigma, t)$ and denote it by F_i. Then the metric tensor is given by

$$g_{ij} = \langle F_i, F_j \rangle$$

Note that

$$\mathrm{Vol}(F(\Sigma, t)) = \int_\Sigma \sqrt{\det(g_{ij}(t))_{ij}} = \int_\Sigma \frac{\sqrt{\det(g_{ij}(t))_{ij}}}{\sqrt{\det(g_{ij}(0))_{ij}}}$$

and the last integrand function above, which we denote by $V(t)$, is independent of the choice of the coordinate system. This is because changing coordinates will give a Jacobian term of the transition function, and since this term will show up on both the numerator and the denominator, they will cancel with each other. Then for every $x \in \Sigma$, we may choose normal coordinates around x such that $g_{ij}(x, 0) = \delta_{ij}$, and thus

$$\dot{\mathrm{Vol}}(F(\Sigma, 0)) = \int_\Sigma \dot{V}(x, 0)$$

Since we have chosen normal coordinates, we have

$$\begin{aligned}\dot{V}(x, 0) &= \frac{1}{2}\frac{d}{dt}(\det(g_{ij}(t))_{ij})\Big|_{t=0} = \frac{1}{2}\mathrm{Tr}(\dot{g_{ij}}(x, 0)) \\ &= \sum_{i=1}^n \langle (F_i)_t, F_i \rangle\Big|_{t=0} = \sum_{i=1}^n \langle \nabla_{F_i} F_t, F_i \rangle\Big|_{t=0} \\ &= \mathrm{div}_\Sigma(F_t)\Big|_{t=0}\end{aligned}$$

We can also relate the above formula to mean curvature of Σ in the following way. If we split $F_t = F_t^\top + F_t^\perp$, then direct computation gives

$$\dot{V}(x, 0) = (-\langle F_t, H \rangle + \mathrm{div}_\Sigma(F_t^\top))\Big|_{t=0}$$

Integrating $\dot{V}(x, 0)$, we have

$$\dot{\mathrm{Vol}}(F(\Sigma, 0)) = \left(-\int_\Sigma \langle F_t, H \rangle \right)\Big|_{t=0} \tag{3.1}$$

where we used the divergence theorem to see that

$$\left(\int_\Sigma \mathrm{div}_\Sigma(F_t^\top) \right)\Big|_{t=0} = 0 \tag{3.2}$$

We now summarize our observations in the following proposition:

Proposition 3.3. *Suppose Σ is a minimal surface and $F(x, t)$ a variation as above. Then*

(1) $\mathrm{div}_\Sigma(F_t)|_{t=0} = 0$

(2) Since such F is arbitrary, we must have H vanish identically on Σ.

In fact, we observe from (3.1) that Σ is a critical point of the volume functional if and only if the mean curvature H vanishes identically on Σ. This motivates us to give one of the equivalent rigorous definitions of minimal submanifolds.

Definition 3.4 (Minimal Submanifolds)**.** An immersed submanifold $\Sigma^k \subset M^n$ is said to be minimal if the mean curvature H vanishes identically.

Remark 3.5. It can be shown that this definition of minimal submanifolds coincides with the definition we gave at the beginning of the article in the case of minimal surfaces in $\mathbb{R}^3$.

Following our computations of the first variation, we introduce some interesting consequences.

Lemma 3.6 (Harmonic Coordinates)**.** *Coordinate functions of a minimal surface Σ are harmonic.*

Proof. We let x_i denote the coordinate functions of Σ. Note that

$$\nabla_\Sigma x_i = \nabla^\top x_i = \partial_i^\top$$

So we have

$$\begin{aligned}\Delta_\Sigma x_i &= \mathrm{div}_\Sigma(\nabla_\Sigma x_i) = \mathrm{div}_\Sigma(\partial_i - \partial_i^\perp)\\ &= -\mathrm{div}_\Sigma(\partial_i^\perp) = \langle H, \partial_i^\perp\rangle = 0\end{aligned}$$

where the last equality is because Σ is minimal. □

Corollary 3.7. *Given any coordinate function u on Σ, u must have its maximum and minimum on the boundary of Σ. Moreover, if Σ is minimal, then $\Sigma \subset \mathrm{Conv}(\partial\Sigma)$. Here Σ is regarded to be embedded in $\mathbb{R}^n$ for some n.*

Proof. The first statement just follows from maximum principle. For the second part, suppose on the contrary that there is a point x that doesn't lie in $\mathrm{Conv}(\partial\Sigma)$. Then there is a hyperplane containing x that doesn't intersect with $\mathrm{Conv}(\partial\Sigma)$. To see this, we note that $\partial\Sigma$ is compact, and thus $\mathrm{Conv}(\partial\Sigma)$ is closed. Then we can choose a plane orthogonal to the projection of x onto $\mathrm{Conv}(\partial\Sigma)$.

We note that $h : \Sigma \to \mathbb{R}$ defined by mapping $y \in \Sigma$ to its distance to the hyperplane can be realized as a coordinate function, and h attains its minimum 0 at x. However, x is not on the boundary $\partial\Sigma$ and we arrive at a contradiction. □

In 1968, Jim Simons gave a fundamental variation formula for minimal surfaces. We consider a map $F : \Sigma \times (-\varepsilon, \varepsilon) \to M^{n+1}$ with the following properties:

(1) Σ is minimal and has trivial normal bundle.
(2) $F(\cdot, 0)$ is the identity map.
(3) F_t has compact support.
(4) $F_t^\top = 0$.

The existence of such an F is guaranteed by the existence and uniqueness of ODE solutions. We shall derive the formula

$$\ddot{V}(0) = -\int_\Sigma \langle F_t, LF_t\rangle \tag{3.8}$$

where if we identify a normal vector field $X = fN$ for some η,

$$L\eta = \Delta_\Sigma f + |A|^2 f + \mathrm{Ric}_M(N)f$$

Here Ric_M is the Ricci curvature on M as seen in [3]. As before, we let x_i be local coordinates on Σ and write

$$g_{ij}(t) = g(F_i, F_j), \quad V(t) := \frac{\sqrt{\det(g_{ij}(t))_{ij}}}{\sqrt{\det(g_{ij}(0))_{ij}}}$$

Then direct computation yields

$$\dot V = \frac{1}{2} V \cdot \mathrm{Tr}(g^{-1}\dot g)$$

and also

$$\ddot V(0) = \frac{1}{2}\dot V \cdot \mathrm{Tr}(g^{-1}\dot g) + \frac{1}{2} V \cdot \mathrm{Tr}(\dot{\overline{g^{-1}}}\dot g) + \frac{1}{2} V \cdot \mathrm{Tr}(g^{-1}\ddot g) \tag{3.9}$$

To better evaluate $\ddot V(0)$ at a specific point, we fix a point x and choose a normal local coordinate around x such that $g_{ij}(x, 0) = \delta_{ij}$. Also we note that $g^{-1}g = \mathrm{Id}$ gives us

$$\dot{\overline{g^{-1}}} = g^{-1}\dot g g^{-1}$$

So (3.9) becomes

$$\ddot V(0) = \frac{1}{2}\mathrm{Tr}(\dot g)^2 + \frac{1}{2}\mathrm{Tr}(\ddot g) = \frac{1}{2}\det(\dot g)^2 + \frac{1}{2}\mathrm{Tr}(\ddot g)$$

where the last inequality utilizes $\mathrm{Tr}(M^2) = \det(M)^2$ for symmetric matrices, which follows from the spectral theorem. Now we need a lemma.

Lemma 3.10. *At the point x, we get*

$$\det(\dot g(0))^2 = 4|\langle A(\cdot,\cdot), F_t\rangle|^2 \tag{3.11}$$

$$\mathrm{Tr}(\ddot g) = 2|\langle A(\cdot,\cdot), F_t\rangle|^2 + 2|\nabla^\perp F_t|^2 - 2\mathrm{Ric}_M(F_t) + 2\mathrm{div}_\Sigma(F_{tt}) \tag{3.12}$$

Proof. (1) Since $g_{ij} = \langle F_i, F_j\rangle$, we have

$$\dot g_{ij} = \langle F_{x_i,t}, F_{x_j}\rangle + \langle F_{x_i}, F_{x_j,t}\rangle$$

Here the notation convention is that

$$F_{x_i,t} = \nabla_{F_t} F_{x_i} = \nabla_{F_t} F_i$$

Since F_t is normal to Σ, the above equation equals to

$$-\langle F_t, \nabla_{F_i} F_j\rangle - \langle F_t, \nabla_{F_j} F_i\rangle = -\langle F_t, \nabla^\perp_{F_i} F_j\rangle - \langle F_t, \nabla^\perp_{F_j} F_i\rangle = -2\langle F_t, A(F_i, F_j)\rangle$$

When $t = 0$, we have

$$\dot g_{ij}(0) = -2\langle F_t, A(F_i, F_j)\rangle$$

Since $(F_i)_i$ at $t = 0$ give an orthonormal basis on $T\Sigma$ at x, we have the desired result.

(2) Note that

$$\mathrm{Tr}(\ddot g(0)) = \sum_i 2\langle F_{x_t,t}, F_{x_i,t}\rangle + 2\langle F_{x_i,tt}, F_{x_i}\rangle$$

Each individual term in the first part is

$$\langle F_{x_t,t}, F_{x_i,t}\rangle = |\nabla_{F_i} F_t|^2 = |\nabla_\Sigma F_t|^2 = |\nabla^\perp_\Sigma F_t|^2 + |\nabla^\top_\Sigma F_t|^2 = |\nabla^\perp_\Sigma F_t|^2 + |\langle A, F_t\rangle|^2$$

and each individual term each the second part is

$$\langle \nabla_{F_t} \nabla_{F_t} F_i, F_i \rangle = \langle \nabla_{F_t} \nabla_{F_i} F_t, F_i \rangle = \langle R(F_i, F_t) F_t, F_i \rangle + \langle \nabla_{F_i} \nabla_{F_t} F_t, F_i \rangle$$

So summing in i gives us the desired result.

□

We are now ready to prove the variation formula. Note that $F_t = fN$ for some smooth function f on Σ, and N is the normal vector field on Σ. For convenience, we use T to denote F_t. Since $\{F_i\}_i$ form an orthonormal frame near x, we have

$$\begin{aligned} \mathrm{div}_\Sigma(F_{tt}) &= \sum_i \langle \nabla_{F_i} \nabla_T T, F_i \rangle = \sum_i \langle \nabla_{F_i} (f \cdot N(f)) N, F_i \rangle \\ &= \sum_i \langle (f \cdot N(f)) \nabla_{F_i} N, F_i \rangle + \langle F_i (f \cdot N(f)) \cdot N, F_i \rangle \\ &= (f \cdot N(f)) \sum_i \langle \nabla_{F_i} N, F_i \rangle \\ &= (f \cdot N(f)) H \end{aligned}$$

The minimality of Σ ensures that H must vanish identically on Σ, so we actually have $\mathrm{div}_\Sigma(F_{tt}) = 0$, and

$$\ddot{V}(0) = -\int_\Sigma |\langle A(\cdot, \cdot), F_t \rangle|^2 - |\nabla_\Sigma^\perp F_t|^2 + \mathrm{Ric}_M(F_t) = -\int_\Sigma \langle F_t, LF_t \rangle$$

as desired.

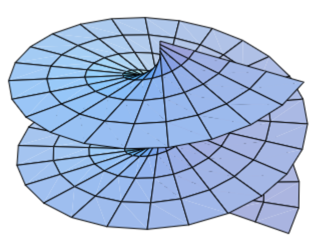
4. Examples of Minimal Submanifolds

In this section, we will provide some examples of minimal Submanifolds. We shall focus on 1 and 2 dimensional manifolds.

Example 4.1 (Geodesics)**.** In 1 dimension, the definition of minimal submanifolds exactly give us geodesics.

Example 4.2 (The Helicoid)**.** The helicoid is given as the set $x_3 = \tan^{-1}(\frac{x_2}{x_1})$. Alternatively it is given in parametric form by

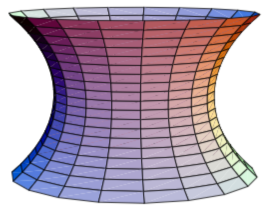
$$(x_1, x_2, x_3) = (t \cos s, t \sin s, s) \quad t, s \in \mathbb{R}$$



It was discovered by Meusnier (a student of Monge) in 1776.

Example 4.3 (The Catenoid)**.** The catenoid is the only nonflat minimal surface of revolution. It is given as the set

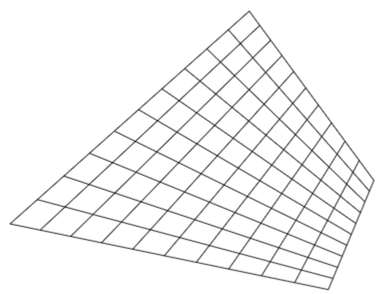
$$\{(x_1, x_2, x_3) : x_1^2 + x_2^2 = \cosh^2(x_3)\}$$



It was disovered by Euler in 1744 and shown to be minimal by Meusnier (a student of Monge) in 1776.

Example 4.4 (Skew Quadrilateral)**.** A skew quadrilateral is a four-sided quadrilateral not contained in a plane. The problem of finding the minimum bounding surface of a skew quadrilateral was solved by Schwarz in terms of Abelian integrals and has the shape of a saddle. It is given by solving

$$(1+f_y^2)f_{xx} - 2f_xf_yf_{xy} + (1+f_x^2)f_{yy} = 0.$$



In general, finding specific examples of minimal surfaces is highly non-trivial and is still a heated topic of ongoing research.

5. Stability of Minimal Surfaces

We now talk about the stability of minimal surfaces, which is closely related to the second derivative of the volume functional.

Definition 5.1. We say that a minimal surface Σ is stable if $\ddot{V}(0) \geqslant 0$ for any compactly supported variation F.

Now, suppose Σ is stable and has trivial normal bundle, and we consider a variation F as in the second half of the previous section. Since Σ is stable, the second variation formula implies

$$\int_\Sigma f \cdot Lf \leqslant 0 \tag{5.2}$$

Here we assume $F_t = fN$. Expanding the definition of L, the above formula is

$$\int_\Sigma f\Delta_\Sigma f + \mathrm{Ric}_M(N)f^2 + |A|^2 f^2 \leqslant 0$$

Integration by parts of the first term gives

$$\int_\Sigma (|A|^2 + \mathrm{Ric}_M(N))f^2 \leqslant \int_\Sigma |\nabla_\Sigma f|^2 \tag{5.3}$$

Suppose that Σ is compact, we may choose $f \equiv 1$, since we know we can find a variation F such that $F_t = N$ by ODE theory. Then (5.3) becomes

$$\int_\Sigma (|A|^2 + \mathrm{Ric}_M(N)) \leqslant 0 \tag{5.4}$$

For an ambient manifold with Ricci curvature strictly greater than 0, we know by (5.4) that there is no compact stable minimal surface with trivial normal bundle.

Now, let L be the same operator as defined above. If L is related to some stable minimal surface, then we also say L is stable.

Lemma 5.5. *L is stable if there is a function $u > 0$ on Σ such that $Lu = 0$.*

Proof. Let f be compactly supported on Σ, and u be a function satisfying the assumption in the problem. We have

$$\mathrm{div}\left(f^2\frac{\nabla u}{u}\right) = 2f\langle\nabla f, \frac{\Delta u}{u}\rangle - f^2\frac{|\nabla u|^2}{u^2}$$

Define $q := |A|^2 + \mathrm{Ric}_M(n)$. Since $Lu = 0$, we have

$$\frac{\Delta u}{u} = -q$$

and thus

$$\begin{aligned}\int_\Sigma f^2 q &= -\int_\Sigma f^2\frac{|\nabla u|^2}{u^2} + 2f\langle\nabla f, \frac{\nabla u}{u}\rangle \\ &= \int_\Sigma \left|\frac{f\nabla u}{u} - \nabla f\right|^2 + |\nabla f|^2 \\ &\leqslant \int_\Sigma |\nabla f|^2\end{aligned}$$

which is equivalent to the stability condition if we use integration by parts. □

As a corollary of the lemma, we can show that minimal graphs in $\mathbb{R}^3$ are always stable.

Corollary 5.6. *Minimal graphs in $\mathbb{R}^3$ are stable.*

Proof. Let ∂z be the constant normal vector field pointing to the z-direction of $\mathbb{R}^3$, and we define $f := \langle\partial z, N\rangle$, where N is the outward normal of the graph Σ. Since Σ is a graph, we must have $f > 0$. By the above lemma, we are done if we can show that $Lf = 0$. We now fix an $x \in \Sigma$ and choose a local geodesic frame $\{x_i\}_i$ around x. Now

$$\begin{aligned}\Delta_\Sigma f &= \sum_i \nabla_{\partial_i}\nabla_{\partial_i}\langle\partial z, N\rangle = \sum_i\langle\partial z, \nabla_{\partial_i}\nabla_{\partial_i}N\rangle = \sum_i\sum_j\langle\partial z, \nabla_{\partial_i}(-A_{ji}\partial_j)\rangle \\ &= \sum_i\sum_j\langle\partial z, -(A_{ji,i})\partial_j\rangle = \sum_i\sum_j\langle\partial z, -A_{ji}\nabla_{\partial_i}\partial_j\rangle\end{aligned} \tag{5.7}$$

It follows from Bianchi identity that $A_{ji,i} = A_{ii,j}$ and we omit the computation details here. Since mean curvature identically vanish on Σ minimal, $A_{ii,j} = 0$. Since we selected geodesic frame around x, we have $\nabla^{\Sigma}_{\partial_i}\partial_j = 0$ and thus $\nabla_{\partial_i}\partial_j = A(\partial_i, \partial_j)$. Continuing with (5.7), we have

$$\Delta_\Sigma f = -|A|^2\langle \partial z, N\rangle = -|A|^2 f$$

Note that the ambient manifold $\mathbb{R}^3$ is flat, so we don't need to consider the Ricci curvature term of L. Thus we have already obtained the desired result. □

6. Simons Inequality

In this section we introduce some Simons inequalities, which illustrate some important techniques in minimal surface theory. They also get used very often in curvature estimates. The results we want to show are the following:

Theorem 6.1. *Suppose Σ^n is a minimal hypersurface of $\mathbb{R}^{n+1}$. Then*

$$\Delta_\Sigma A = -|A|^2 A$$

In particular, we have

$$\frac{1}{2}\Delta_\Sigma |A|^2 = |A|^4 + |\nabla A|^2$$

and thus

$$\frac{1}{2}\Delta_\Sigma |A|^2 \geqslant -|A|^4$$

The key ingredient of the proof are the Gauss and Codazzi equations.

Theorem 6.2 (Chapter 6 of [3])**.** *Let $M \subset \overline{M}$ be smooth manifolds, and $\overline{R}$ be the Riemannian curvature tensor on $\overline{M}$.*

(1) (Gauss Equation) Let X, Y, Z, T be vector fields on $\overline{M}$ that are tangent to M, then

$$\langle \overline{R}(X,Y)Z, T\rangle = \langle R(X,Y)Z,T\rangle - \langle A(Y,T), A(X,Z)\rangle + \langle A(X,T), A(Y,Z)\rangle$$

(2) (Codazzi Equation) Let X, Y, Z be vector fields on $\overline{M}$ that are tangent to M, and η a vector on $\overline{M}$ that is normal to M. Then

$$\langle \overline{R}(X,Y)Z, \eta) = X(\langle A(Y,Z), \eta\rangle) - \langle A(\nabla_X Y, Z), \eta\rangle$$

In our context, Σ is a hypersurface of $\mathbb{R}^{n+1}$, and the ambient manifold has $\overline{R} = 0$. The Gauss equation becomes

$$R_{ijkl} = A_{ik}A_{jl} - A_{jk}A_{il}$$

and the Codazzi equation becomes

$$A_{jk,i} = A_{ik,j}$$

We also state a lemma here, whose proof is direct computation.

Lemma 6.3. *We have*

$$A_{ij,jk} - A_{ij,kl} = \sum_m R_{lkim}A_{mj} + R_{lkjm}A_{mi}$$

where R denotes the Riemannian curvature tensor on Σ.

With all these tools in hand we are now ready to prove the Theorem.

Proof of Theorem 6.1. We fix a point x and let choose a geodesic frame around x. Then

$$\Delta_\Sigma A(\partial_i, \partial_j) = \sum_k A_{ij,kk}$$

By Codazzi equation and the lemma, the above equation is equal to

$$\sum_k A_{ik,jk} = \sum_k A_{kk,ij} + \sum_{k,m} R_{kjim} A_{mk} + R_{kjkm} A_{mi}$$

The first summand above on the right hand side above gives the Hessian of mean curvature, and the second summand gives the rest of the desired terms in our conclusion by Gauss equation. □

Note that mean curvature vanishes identically if Σ is minimal. If we apply Theorem 6.1 to Σ minimal, the Hessian term vanishes and we get the Simons equality. Now our proof is complete.

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