NOTES ON HARMONIC ANALYSIS

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Abstract. These notes record some foundational knowledge for PDE research and are evolving over time. They are supposed to be high-level sketches, so intuitions and main ideas are emphasized, and sometimes only references are given for detailed proofs. At the meantime, examples are computed, and proofs not easily found in literature are supplied. (Last Updated: Jan 22, 2024.)

Contents

1. Two Fundamental Interpolation Theorems

We begin with the Marcinkiewicz interpolation theorem, which sometimes is also called "real interpolation."

Theorem 1.1 (Marcinkiewicz Interpolation)**.** *Let X and Y be measurable spaces, and T a sublinear operator that takes a dense subset of measurable functions on X to measurable functions on Y. For $1 \leqslant p \leqslant q \leqslant \infty$, if T is weak (p,p) and weak (q,q), then T is strong (r,r) for all $p < r < q$.*

The proof crucially replies on layer-cake decomposition of integrals

$$\int |g|^p \, d\mu = \int \lambda^{p-1} \mu(\{|g| > \lambda\}) \, d\lambda$$

and the decomposition

$$Tf = T(f 1_{\{|f| \leqslant \lambda\}} + f 1_{\{|f| > \lambda\}}) \leqslant T(f 1_{\{|f| \leqslant \lambda\}}) + T(f 1_{\{|f| > \lambda\}})$$

For a slightly more general version of Marcinkiewicz interpolation and a proof, see [4].

Theorem 1.2 (Riesz-Thorin Interpolation)**.** *Let X and Y be measurable spaces, and T a sublinear operator that takes a dense subset of measurable functions on X to measurable functions on Y. Let $1 \leqslant p_0, q_0, p_1, q_1 \leqslant \infty$. If*

$$\|T\|_{L^{p_0} \to L^{q_0}} \leqslant k_0, \quad \|T\|_{L^{p_1} \to L^{q_1}} \leqslant k_1$$

then T is bounded $L^{p_\theta} \to L^{q_\theta}$ for

$$\frac{1}{p_\theta} = \frac{\theta}{p_0} + \frac{1-\theta}{q_0}, \quad \frac{1}{q_\theta} = \frac{\theta}{p_1} + \frac{1-\theta}{q_1}$$

with

$$\|T\|_{L^{p_\theta} \to L^{q_\theta}} \leqslant k_0^\theta k_1^{1-\theta}$$

The proof of Riesz-Thorin (as can be found in [4]) relies on the Three-Lines Lemma in complex analysis:

Lemma 1.3. *We define $S := \{0 < \mathrm{Re}(z) < 1\}$, $S_0 := \{\mathrm{Re}(z) = 0\}$, and $S_1 := \{\mathrm{Re}(z) = 1\}$. Suppose F is continuous on $\overline{S}$ and analytic on S, with*

$$\sup_{S_0} F \leqslant k_0, \quad \sup_{S_1} F \leqslant k_1$$

Then for $x + iy \in S$ we have

$$F(x+iy) \leqslant k_0^{1-x} k_1^x$$

In fact, the proof is robust and can be extended to multilinear settings. As an example, we present the following proposition.

Proposition 1.4 (Bilinear Riesz-Thorin)**.** *Let $B(f,g)$ be a bilinear map in f and g. Assume that there is a constant $c > 0$ such that*

$$\|B(f,g)\|_{L^2} \leqslant c\|f\|_{L^\infty}\|g\|_{L^2}$$

and

$$\|B(f,g)\|_{L^2} \leqslant c\|f\|_{L^2}\|g\|_{L^\infty}$$

Then for every $p, q > 1$ satisfying $\frac{1}{p} + \frac{1}{q} = \frac{1}{2}$, there is a constant $c' > 0$, independent of f and g, such that

$$\|B(f,g)\|_{L^2} \leqslant c'\|f\|_{L^p}\|g\|_{L^q}$$

Proof. For convenience, we suppose f is a measurable function on X, g a measurable function on Y, and $B(f,g)$ a measurable function on Z. As in the proof of Riesz-Thorin, it suffices to show the result for simple functions, which are dense in every L^p $(1 \leqslant p \leqslant \infty)$. Let Σ_X, Σ_Y, Σ_Z be the space of simple functions on X, Y, and Z, respectively. By duality, we wish to establish the following claim:

- If $f \in \Sigma_X$, $g \in \Sigma_Y$ are such that $\|f\|_{L^p} = \|g\|_{L^q} = 1$, then for any $h \in \Sigma_Z$ such that $\|h\|_{L^2} = 1$, we have
$$\left| \int B(f,g)h \right| \leqslant c$$

To show the claim, we begin by assuming that f, g, and h have form $f = \sum_1^m a_j 1_{E_j}$, $g = \sum_1^n b_j 1_{F_j}$, $h = \sum_1^r c_j 1_{G_j}$, where F_j's are disjoint, E_j's are disjoint, and F_j's are disjoint, and a_j, b_j, c_j's are non-zero. This can be achieved by writing the functions as their standard representations. Then we can write $a_j = |a_j|e^{i\theta_j}$, $b_j = |b_j|e^{i\varphi_j}$, and $c_j = |c_j|e^{i\psi_j}$, in their polar forms. For $0 < t < 1$, we define

$$\frac{1}{p_t} = \frac{t}{2}, \quad \frac{1}{q_t} = \frac{1-t}{2}$$

where $\|f\|_{L^{p_t}} = \|g\|_{L^{q_t}} = 1$. We also define

$$f_z := \sum_{j=1}^n |a_j|^{z/t} e^{i\theta_j} 1_{E_j}, \quad g_z := \sum_{j=1}^m |b_j|^{\frac{1-z}{1-t}} e^{i\varphi_j} 1_{F_j}$$

$$\phi(z) := \int B(f_z, g_z)h = \sum_{j,k,l} A_{j,k,l} |a_j|^{z/t} |b_k|^{\frac{1-z}{1-t}} |c_l|$$

where

$$A_{jkl} = e^{i(\theta_j + \varphi_k + \psi_l)} \int B(1_{E_j}, 1_{F_k}) 1_{G_k}$$

Thus $\phi(z)$ is an entire function of z bounded in the strip $0 \leqslant \mathrm{Re} z \leqslant 1$. Notice that

$$\left| \int B(f,g)h \right| = \phi(t)$$

So by the Hadamard three lines lemma it suffices to show that $|\phi(z)| \leqslant c$ for $\mathrm{Re} z = 0$ and $\mathrm{Re} z = 1$. Now, note that for $s \in \mathbb{R}$,

$$|f_{is}| = \sum_{j=1}^n |a_j|^{\mathrm{Re} is/t} 1_{E_j} \leqslant 1$$

$$|g_{is}| = \sum_{j=1}^n |a_j|^{\mathrm{Re}\frac{1-is}{1-t}} 1_{E_j} = \sum_{j=1}^n |a_j|^{\frac{1}{1-t}} 1_{E_j}$$

and thus

$$|\phi(is)| \leqslant \|B(f_{is}, g_{is})\|_{L^2} \|h\|_{L^2} \leqslant c\|f_{is}\|_{L^\infty} \|g_{is}\|_{L^2} \leqslant c\|g_{is}\|_{L^{q_t}}^{\frac{1}{1-t}} = c$$

Similarly we can calculate

$$|\phi(1+is)| \leqslant c$$

and thus we have shown the claim as desired. ∎

2. Maximal Functions

We introduce the Calderón-Zygmund decomposition. Let $f \in L^1_{loc}(\mathbb{R}^n)$ and $\lambda > 0$. We define

$$\Omega_\lambda := \{M_d f > \lambda\}$$

and we can write

$$\Omega_\lambda = \cup_j Q_j$$

where each Q_j is a maximal dyadic cube. We note that in fact

$$\frac{1}{|Q_j|}\int_{Q_j} |f(x)|\, dx \geqslant \lambda$$

because at every $x \in \Omega_\lambda$ one can find a small enough cube on which the average of $|f|$ is greater than λ. Moreover, if $\tilde{Q}_j$ is the parent of Q_j, we must have

$$\frac{1}{|\tilde{Q}_j|}\int_{\tilde{Q}_j} |f(x)|\, dx < \lambda$$

otherwise there would be some $y \in \tilde{Q}_j \backslash \Omega_\lambda$ having

$$M_d f(y) \geqslant \frac{1}{|\tilde{Q}_j|}\int_{\tilde{Q}_j} |f(x)|\, dx \geqslant \lambda$$

contradicting the fact that $y \notin \Omega_\lambda$. In summary we have

$$\lambda \leqslant \frac{1}{|Q_j|}\int_{Q_j} |f(x)|\, dx \leqslant \frac{1}{|Q_j|}\int_{\tilde{Q}_j} |f(x)| \leqslant 2^n \lambda$$

This motivates the following definition/theorem:

Theorem 2.1 (Calderón-Zygmund Decomposition)**.** *Given $f \in L^1_{loc}(\mathbb{R}^n)$ and $\lambda > 0$, we can find a decomposition $f = g + b$ and a collection of mutually disjoint dyadic cubes $\{Q_j\}_j$, such that $|g| \leqslant 2^n \lambda$ a.e., $m_{Q_j} b = 0$ for every j, and*

$$\frac{1}{Q_j}\int_{Q_j} |b(x)|\, dx \leqslant 2^{n+1}\lambda$$

Proof. We define

$$g(x) := \begin{cases} f(x) & x \notin \Omega_\lambda \\ \frac{1}{|Q_j|}\int_{Q_j} f(x)\, dx & x \in Q_j \end{cases}$$

and $b = f - g$. According to our discussions above, this decomposition will do the job. □

3. BMO Spaces

We will see that BMO is a natural extension of L^∞ in many situations.

In this section, Q will be assumed to denote cubes in $\mathbb{R}^n$ unless otherwise stated. Without special notices, $\sup_Q$ means supremum over all cubes in $\mathbb{R}^n$.

Definition 3.1. For $f \in L^1_{loc}(\mathbb{R}^n)$, We define

$$m_Q f := \frac{1}{|Q|}\int_Q f(x)\, dx \tag{3.2}$$

to be the average of f over Q, and the **mean oscillation**

$$\|f\|_* := \sup_Q \frac{1}{|Q|}\int_Q |f - m_Q f|\ dx \tag{3.3}$$

If $\|f\|_*$ is finite, we say that f has **bounded mean oscillation**, or more concisely that f is a BMO function.

The natural next step is to look at the collection of BMO functions, and see whether $\|\cdot\|_*$ is a well-defined norm that makes BMO a Banach space. However, we note that $\|\cdot\|_*$ is only a seminorm, since $\|f + C\|_* = \|f\|_*$ for any constant C. Fortunately this is the only issue to remedy, and we make the following definition:

Definition 3.4 (BMO)**.** We define the equivalence relation $f \sim g$ iff $f - g$ is a constant, and let $[f]$ be the equivalence class of f. We define the normed linear space

$$\mathrm{BMO}(\mathbb{R}^d) := \{[f] : f : \mathbb{R}^d \to \mathbb{C} \text{ is a BMO function}\} \tag{3.5}$$

equipped with the norm

$$\|[f]\|_* := \sup_Q \frac{1}{|Q|}\int_Q |f - m_Q f|\ dx \tag{3.6}$$

Remark 3.7. (1) It can be seen that the norm (3.6) is well-defined.
(2) We often abuse notation, and write $[f]$ and f interchangeably.

Proposition 3.8 (Equivalent BMO Norms)**.** *We have the following equivalent characterizations of BMO norm:*

(1) We have

$$\frac{1}{2}\|f\|_* \leqslant \sup_Q \inf_{\alpha\in\mathbb{R}} \frac{1}{|Q|}\int_Q |f(x) - \alpha|\ dx \leqslant 2\|f\|_*$$

(2) Moreover,

$$\|f\|_* \leqslant \sup_Q \frac{1}{|Q|^2}\int_Q\int_Q |f(x) - f(y)|\ dxdy \leqslant 2\|f\|_*$$

Proof. (1) The second inequality is immediate. For the first inequality, notice that

$$\begin{aligned}
\int_Q |f - m_Q f| &\leqslant \int_Q |f - \alpha| + \int_Q |\alpha - m_Q f| \\
&\leqslant \int_Q |f - \alpha| + \left|\int_Q \alpha - f\right| \\
&\leqslant 2\int_Q |f - \alpha|
\end{aligned}$$

Dividing both sides by $|Q|$ and taking infimum over all α we get the desired result.
(2) Noticing that

$$|f(x) - f(y)| \leqslant |f(x) - m_Q f| + |f(y) - m_Q f|$$

we easily obtain the second inequality. For the first inequality, note that

$$\begin{aligned}\frac{1}{|Q|}\int_Q |f - m_Q f| &= \frac{1}{|Q|}\int_Q \left| f(x) - \frac{1}{|Q|}\int_Q f(y)\ dy \right| dx \\ &\leqslant \frac{1}{|Q|^2}\int_Q\int_Q |f(x) - f(y)|\ dydx\end{aligned}$$

□

Proposition 3.9. *We have the following properties of BMO:*

(1) $(\mathrm{BMO}, \|\cdot\|_*)$ *is a Banach space.*

(2) $\log|x|$ *is a BMO function. In particular,* L^∞ *is a proper subset of* BMO.

Proof. (1) We only show completeness here. Let $\{f_n\}_n$ be a Cauchy sequence in BMO, and $Q \subset \mathbb{R}^n$ be a cube. Then

$$\begin{aligned}\frac{1}{|Q|}\int_Q |(f_n - m_Q f_n) - (f_m - m_Q f_m)|\ dx &= \frac{1}{|Q|}\int_Q |(f_n - f_m) - m_Q(f_n - f_m)|\ dx \\ &= \|f_n - f_m\|_{BMO}\end{aligned}$$

showing that $\{f_n - m_Q f_n\}_n$ is a Cauchy sequence in $L^1(Q)$. By completeness of $L^1(Q)$, $\{f_n - m_Q f_n\}_n$ has a limit f_Q. Moreover, for $Q_1 \subset Q_2$, we have

$$\begin{aligned}f_{Q_1} - f_{Q_2} &= \lim_{n\to\infty}(f_n - m_{Q_1} f_n) - (f_n - m_{Q_2} f_n) \\ &= \lim_{n\to\infty} m_{Q_2} f_n - m_{Q_1} f_n\end{aligned}$$

on Q_1. We know the last line of the above equation is a constant and we define it by $C(Q_1, Q_2)$. We could also see that the constants have transitivity. That is, for $Q_1 \subset Q_2 \subset Q_3$,

$$C(Q_1, Q_3) = C(Q_1, Q_2) + C(Q_2, Q_3)$$

Now, we define Q_k as the cube centered at the origin and has radius k, where k is a natural number. Then the sequence $\{Q_k\}_k$ exhausts $\mathbb{R}^n$, meaning that $\bigcup_k Q_k = \mathbb{R}^n$ and $Q_{k_1} \subset Q_{k_2}$ for $k_1 \leqslant k_2$. We define a function f on $\mathbb{R}^n$ such that

$$f = f_{Q_k} + C(Q_1, Q_k)$$

on Q_k.

First of all we want to see that f is well-defined. If $Q_k \subset Q_l$, we have

$$f_{Q_k} - f_{Q_l} = C(Q_k, Q_l) = C(Q_1, Q_l) - C(Q_1, Q_k)$$

and thus

$$f_{Q_k} + C(Q_1, Q_k) = f_{Q_l} + C(Q_1, Q_l)$$

so different representations of f are equal. We then want to see that $f_n \to f$ in BMO. Let Q be a cube. Then $Q \subset Q_k$ for some k large enough, and

$$\frac{1}{|Q|}\int_Q |f_n - m_Q(f_n) - (f - m_Q(f))| = \frac{1}{|Q|}\int_Q |f_n - m_Q(f_n) - (f_{Q_k} - m_Q(f_{Q_k}))|$$

Note that

$$f_n - m_Q(f_n) - f_{Q_k} + m_Q(f_{Q_k}) = (f_n - m_Q(f_n) - f_Q) + (f_Q - f_{Q_k} + m_Q(f_{Q_k}))$$

and also

$$\begin{aligned} f_Q - f_{Q_k} + m_Q(f_{Q_k}) &= \lim_{n\to\infty}(m_{Q_k}(f_n) - m_Q(f_n)) + \lim_{n\to\infty} m_Q(f_n - m_{Q_k}(f_n)) \\ &= \lim_{n\to\infty}(m_{Q_k}(f_n) - m_Q(f_n) + m_Q(f_n) - m_{Q_k}(f_n)) \\ &= 0 \end{aligned}$$

So we have

$$\lim_{n\to\infty}\frac{1}{|Q|}\int_Q |f_n - m_Q(f_n) - (f - m_Q(f))| = \lim_{n\to\infty}\frac{1}{|Q|}\int_Q |f_n - m_Q(f_n) - f_Q|\, dx = 0$$

by our definition of f_Q.

Note that Q is arbitrary, and we claim that in fact $f_n \to f$ in BMO. Suppose on the contrary that f_n doesn't converge to f in BMO, then there is a $\delta > 0$, a sequence of indices $n_k \in \mathbb{N}$, and a sequence of cubes $Q_{n_k} \subset \mathbb{R}^n$ such that

$$\frac{1}{|Q_{n_k}|}\int_{Q_{n_k}} |f_{n_k} - f - m_Q(f_{n_k} - f)|\, dx \geqslant \delta$$

However, for every fixed n_k, we have

$$\lim_{n\to\infty}\frac{1}{|Q_{n_k}|}\int_{Q_{n_k}} |f_n - f - m_Q(f_n - f)|\, dx = 0$$

So if we extract the diagonal we should also have

$$\lim_{k\to\infty}\frac{1}{|Q_{n_k}|}\int_{Q_{n_k}} |f_{n_k} - f - m_Q(f_{n_k} - f)|\, dx = 0$$

which gives us a contradiction. Hence our claim is proven, and BMO is indeed complete.

(2) By Proposition 3.8, we just need to show that there is a universal constant C such that for every $B(x_0, R) \subset \mathbb{R}^n$, we have

$$\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} |\log|x| - \alpha|\, dx \leqslant C$$

for some α depending on x_0 and R. As a first step, we do a change of variables to arrive at

$$\frac{1}{|B(x_0,R)|}\int_{B(x_0,R)} |\log|x| - \alpha|\, dx = \int_{B(0,1)} |\log|x_0 + Rx| - \alpha|\, dx$$

Suppose $|x_0| \leqslant 2R$. Then we note that for $x \in B(0,1)$,

$$|\log|x_0 + Rx| - \log R| = \left|\log\left|x + \frac{x_0}{R}\right|\right|$$

and thus

$$\begin{aligned} \int_{B(0,1)} |\log|x_0 + Rx| - \log R| &= \int_{B(0,1)} \left|\log\left|x + \frac{x_0}{R}\right|\right| = \int_{B(-x_0/R,3)} \left|\log\left|x + \frac{x_0}{R}\right|\right| \\ &= \int_{B(0,3)} |\log|x|| < \infty \end{aligned}$$

For $|x_0| > 2R$, we note that for $x \in B(0,1)$,

$$| \log |x_0 + Rx| - \log(|x_0|/2)| \leqslant \left| \log \left(2 + \frac{2R|x|}{|x_0|} \right) \right| \leqslant \log 3$$

and thus

$$\int_{B(0,1)} | \log |x_0 + Rx| - \log(|x_0|/2)| \, dx \leqslant \max \left(\log 3, \int_{B(0,3)} | \log |x|| \right)$$

so we have the desired conclusion.

□

Given a BMO function, we can actually construct a sequence of L^∞ functions to approximate it locally in L^1.

Proposition 3.10. *Define*

$$f_q(x) = \begin{cases} f(x) & |f(x)| \leqslant q \\ \frac{f(x)}{|f(x)|} q & |f(x)| \geqslant q \end{cases}$$

for $q > 0$. Then

(1) $f_q \in L^\infty$ with $\|f_q\|_{L^\infty} \leqslant q$.
(2) $\|f_q\|_* \leqslant \|f\|_*$
(3) $f_q \to f$ locally in L^1 as $q \to \infty$.

Proof. We only prove (2). Note that for every x, y we have

$$|f_q(x) - f_q(y)| \leqslant |f(x) - f(y)|$$

so for any cube Q we have

$$\frac{1}{|Q|^2} \int_Q \int_Q |f_q(x) - f_q(y)| \, dxdy \leqslant \frac{1}{|Q|^2} \int_Q \int_Q |f(x) - f(y)| \, dxdy$$

Taking supremum over Q we get the desired result. □

We now introduce the important John-Nirenberg inequality.

Theorem 3.11 (John-Nirenberg Inequality)**.** *There are universal constants C and λ such that*

$$\sup_Q \frac{1}{|Q|} \int_Q \exp \left(\frac{\lambda}{\|f\|_*} |f - m_Q f| \right) dx \leqslant C \tag{3.12}$$

Equivalently, there are universal constants C_0 and λ_0 such that

$$|\{x \in Q : |f - m_Q f| > t\|f\|_*\}| \leqslant C_0 e^{-\lambda_0 t} |Q| \tag{3.13}$$

Let's see how (3.12) implies (3.13). Note that

$$\begin{aligned} C|Q| &\geqslant \int_Q \exp \left(\frac{\lambda}{\|f\|_*} |f - m_Q f| \right) dx \\ &\geqslant \int_{Q \cap \{|f - m_Q f| > t\|f\|_*\}} e^{\lambda t} \, dx \\ &= e^{\lambda t} |\{x \in Q : |f - m_Q f| > t\|f\|_*\}| \end{aligned} \tag{3.14}$$

as desired. Conversely,

$$\begin{aligned}\int_Q \exp\left(-\frac{\lambda}{\|f\|_*}|f-m_Qf|\right)\,dx &= \int_0^\infty \lambda e^{-\lambda\mu}|\{x\in Q: |f(x)-m_Qf|>\mu\|f\|_*\}|\,d\mu \\ &\leqslant C_0\int_0^\infty e^{-(\lambda_0+\lambda)\mu}\,d\mu \\ &\leqslant C\end{aligned} \tag{3.15}$$

for some constant $C>0$.

Remember that in Proposition 3.9 we stated that $\log|x|$ is a typical BMO function that is not L^∞. One important heuristic implication of John-Nirenberg is that logarithmic growth is the most one can have in BMO space. To see this, consider $f(x):=\log(1/|x|)$. Then on the interval $Q:=(-a,a)$, $m_Qf=1-\log a$. We compute that for $\lambda>1$,

$$|\{x\in Q: |f-m_Qf|>\lambda\}| = 2ae^{-\lambda-1} = e^{-\lambda-1}|Q|$$

which is the rate given by John-Nirenberg.

Now we prove the John-Nirenberg inequality. The key ingredient of the proof is a Calderon-Zygmund type decomposition.

Proof. First of all we claim that it suffices to assume $f\in L^\infty$ by the above proposition. Suppose we can establish the result for L^∞ functions, then there exist $\lambda, C>0$ such that

$$\sup_Q \frac{1}{|Q|}\int_Q \exp\left(\frac{\lambda}{\|f\|_*}|f_q(x)-m_Qf_q|\right)\,dx \leqslant C$$

Sending $q\to\infty$ and using Fatou's lemma we obtain the desired inequality.

Let $Q_0\subset\mathbb{R}^n$ be a cube, and we use $\Delta(Q_0)$ to denote the collection of dyadic cubes with respect to Q_0. Define $g:=(f-m_{Q_0}f)1_{Q_0}$, and

$$S:=\{Mg>2\|f\|_*\}$$

We know we can write $S=\bigcup_j Q_j$ as a union of maximal dyadic cubes. For every such maximal dyadic cube Q_j, we know that

$$\frac{1}{|Q_j|}\int_{Q_j}|g|\geqslant 2\|f\|_*, \quad \frac{1}{|\tilde{Q}_j|}\int_{\tilde{Q}_j}|g|\leqslant 2\|f\|_*$$

where $\tilde{Q}$ is the parent cube of Q. Now,

$$\begin{aligned}\int_{Q_0}\exp\left(\frac{\lambda}{\|f\|_*}|f-m_{Q_0}f|\right) &= \int_{Q_0\cap S}\exp\left(\frac{\lambda}{\|f\|_*}|f-m_{Q_0}f|\right)+\int_{Q_0-S}\exp\left(\frac{\lambda}{\|f\|_*}|f-m_{Q_0}f|\right) \\ &\leqslant \int_{Q_0\cap S}\exp\left(\frac{\lambda}{\|f\|_*}|f-m_{Q_0}f|\right)+e^{2\lambda}|Q_0|\end{aligned}$$

and if we denote

$$X(\lambda):=\sup_Q\frac{1}{|Q|}\int_Q\exp\left(\frac{\lambda}{\|f\|_*}|f-m_Qf|\right)$$

we have

$$\begin{aligned}\int_{Q_0\cap S}\exp\left(\frac{\lambda}{\|f\|_*}|f-m_{Q_0}f|\right) &= \sum_j\int_{Q_0\cap Q_j}\exp\left(\frac{\lambda}{\|f\|_*}|f-m_{Q_0}f|\right)\\ &\leqslant \sum_{Q_j\cap Q_0\neq\varnothing}|Q_j|\frac{1}{|Q_j|}\int_{Q_j}\exp\left(\frac{\lambda}{\|f\|_*}|f-m_{Q_0}f|\right)\end{aligned}$$

We decompose

$$\int_{Q_j}\exp\left(\frac{\lambda}{\|f\|_*}|f-m_{Q_0}f|\right)=\exp\left(\frac{\lambda}{\|f\|_*}|m_{Q_0}f-m_{Q_j}f|\right)\int_{Q_j}\exp\left(\frac{\lambda}{\|f\|_*}|f-m_{Q_j}f|\right)$$

Note that

$$\begin{aligned}|m_{Q_0}f-m_{Q_j}f| &\leqslant \frac{1}{|Q_j|}\int_{Q_j}|f-m_{Q_0}f|\cdot 1_{Q_0}\\ &\leqslant \frac{2^n}{|\tilde{Q}_j|}\int_{\tilde{Q}_j}|f-m_{Q_0}f|\\ &\leqslant 2^{n+1}\|f\|_*\end{aligned}$$

and that

$$\sum_j|Q_j|\leqslant\frac{1}{2\|f\|_*}\int_{\cup_j Q_j}|g|\leqslant\frac{|Q_0|}{2}$$

so we have

$$\frac{1}{|Q_0|}\int_{Q_0}\exp\left(\frac{\lambda}{\|f\|_*}|f-m_{Q_0}f|\right)\leqslant\frac{e^{2^{d+1}\lambda}}{2}X(\lambda)+e^{2\lambda}$$

Taking supremum over Q_0, we get the desired result if we take a small enough $\lambda>0$. $\square$

Corollary 3.16. *Let $f\in BMO$. Then for $1\leqslant p<\infty$, $f\in L^p_{loc}(\mathbb{R}^n)$ and there is a constant $C_p>0$ such that*

$$\sup_Q\left(\frac{1}{|Q|}\int_Q|f(x)-m_Qf|^p\,dx\right)^{\frac{1}{p}}\leqslant C_p\|f\|_*$$

Moreover, for $1\leqslant p<\infty$,

$$\|f\|_{*,p}:=\sup_Q\left(\frac{1}{|Q|}\int_Q|f(x)-m_Qf|^p\,dx\right)^{\frac{1}{p}}$$

is a norm on BMO *that is equivalent to $\|\cdot\|_*$.*

The proof of Corollary 3.16, which is essentially a layer-cake decomposition plus John-Nirenberg, can be easily adapted to give the following converse of John-Nirenberg.

Corollary 3.17. *Given a function f, suppose there exist constants C_1, C_2, and K such that for any cube Q and $\lambda>0$,*

$$|\{x\in Q:|f(x)-m_Qf|>\lambda\}|\leqslant C_1e^{-C_2\lambda/K}|Q|$$

Then $f\in$ BMO.

4. A_p Weights

Definition 4.1 (A_p Weights)**.** Let $1 < p < \infty$. A locally integrable function ω is an A_p weight if there is some constant C such that

$$\left(\frac{1}{|Q|}\int_Q \omega\right)\left(\frac{1}{|Q|}\int_Q \omega^{1-p'}\right)^{p-1} \leqslant C$$

for all cubes $Q \subset \mathbb{R}^n$. Here $\frac{1}{p} + \frac{1}{p'} = 1$. We say ω is an A_1 weight if there is a $C > 0$ such that for any cube Q,

$$\frac{\omega(Q)}{|Q|} \leqslant C\omega(x)$$

for a.e. $x \in Q$.

Proposition 4.2. *Suppose f is in BMO, then for sufficiently small $\lambda > 0$, there is some $C > 0$ such that*

$$\frac{1}{|Q|}\int_Q \exp|\lambda f| < C$$

for all cubes Q.

Theorem 4.3. *Let f be a locally integrable function such that* $\exp(f) \in A_2$*, then f is in BMO. Conversely, if f is in BMO, then for sufficiently small $\lambda > 0$,* $\exp(\lambda f)$ *is an A_2 weight.*

5. Hardy Spaces and Relation with BMO

Definition 5.1. Let $1 < q \leqslant \infty$. We say a is a $(1, p)$ atom if there is a cube Q such that

(1) Supp $a \subset Q$.
(2) $a \in L^p(Q)$ with $\|a\|_{L^p(Q)} \leqslant |Q|^{\frac{1}{p}-1}$
(3) a has mean 0, namely $\int_Q a(x)\, dx = 0$

We write the collection of $(1, p)$ atoms as $\mathcal{a}^p$. Note first that any element of $\mathcal{a}^p$ is in $L^1(Q)$, with

$$\int_Q |a(x)|\, dx \leqslant \left(\int_Q |a(x)|^p\right)^{\frac{1}{p}} |Q|^{\frac{1}{p'}} \leqslant 1 \tag{5.2}$$

Moreover, we have by Holder's inequality that $\mathcal{a}^{p_1} \subset \mathcal{a}^{p_2}$ as long as $p_1 < p_2$.

Definition 5.3. Let $1 < p \leqslant \infty$. We say that $f \in H^{1,p}(\mathbb{R}^d)$ if there exists a sequence of $(1, p)$ atoms $\{a_j\}_j$ and a sequence of real numbers $\{\lambda_j\}_j$ such that

(1) $\sum_j \lambda_j < \infty$.
(2) $f(x) = \sum_j \lambda_j a_j$

Moreover, we define

$$\|f\|_{H^{1,p}} := \inf\left\{\sum_j \lambda_j : f(x) = \sum_j \lambda_j a_j\right\}$$

Remark 5.4. (1) $\sum_j \lambda_j a_j$ converges in L^1.
(2) $H^{1,p}$ is Banach if $1 < p \leqslant \infty$.

Proposition 5.5. *For $1 < p \leqslant \infty$, $H^{1,p} = H^{1,\infty}$. We can thus define $H^1 := H^{1,p}$, where $1 < p \leqslant \infty$.*

We have the following important characterization of H^1 and BMO spaces.

Theorem 5.6. $(H^1)^* = \mathrm{BMO}$.

6. An Interpolation Theorem for BMO

Marcinkiewicz interpolation theorem tells us that weak $(1,1)$ boundedness and strong (∞,∞) boundedness together give us strong (p,p) boundedness for all $1 < p < \infty$. In practical applications, the assumption on L^∞ boundedness is somtimes too strong, so in this section we present an interpolation theorem that only assumes $L^\infty_c \to BMO$ boundedness on the infinity end.

Lemma 6.1 (Good-λ-Inequality)**.** *Let $p > 0$. Let $u, v \in L^1_{loc}(\mathbb{R}^n)$, u, v non-negative. Assume that*

(1) $\inf(1,u) \in L^p(\mathbb{R}^n)$.

(2) There exist $\varepsilon > 0$ and $0 < \gamma < 1$ such that

$$|\{x \in \mathbb{R}^n : u(x) > (1+\varepsilon)\lambda,\ v(x) \leqslant \lambda\}| \leqslant \gamma |\{x \in \mathbb{R}^n : u(x) > \lambda\}|$$

Then there exists a constant $C = C(\varepsilon, \gamma, p) > 0$ such that

$$\|u\|_{L^p} \leqslant C\|v\|_{L^p}$$

Proof. We first assume $u \in L^p(\mathbb{R}^n)$. By layer-cake decomposition of the integral,

$$\begin{aligned}
\|u\|_{L^p}^p &= \int_0^\infty \lambda^{p-1}|\{u > \lambda\}|\ d\lambda \\
&= \frac{1}{(1+\varepsilon)^p}\int_0^\infty \lambda^{p-1}|\{u > (1+\varepsilon)\lambda\}|\ d\lambda \\
&\leqslant \frac{1}{(1+\varepsilon)^p}\left(\int_0^\infty \lambda^{p-1}|\{u > (1+\varepsilon)\lambda,\ v \leqslant \lambda\}|\ d\lambda + \int_0^\infty \lambda^{p-1}|\{v > \lambda\}|\ d\lambda\right) \\
&\leqslant \frac{\gamma}{(1+\varepsilon)^p}\|u\|_{L^p}^p + \|v\|_{L^p}^p
\end{aligned}$$

and thus

$$\|u\|_{L^p} \leqslant C(p, \varepsilon, \gamma)\|v\|_{L^p}$$

If only $\inf(1,u) \in L^p$, note that actually $u_n := \inf(n,u) \in L^p$ for every $n \in \mathbb{N}$. We have

$$\|u_n\|_{L^p} \leqslant C(p, \varepsilon, \gamma)\|v\|_{L^p}$$

By Fatou's lemma,

$$\|u\|_{L^p} \leqslant \liminf_{n\to\infty} \|u_n\|_{L^p} \leqslant C(p, \varepsilon, \gamma)\|v\|_{L^p}$$

so we get the desired result. □

Definition 6.2 (Sharp Maximal Function)**.** Let $f \in L^1_{loc}(\mathbb{R}^n)$. We define

$$M^\# f(x) := \sup_{Q \ni x} \frac{1}{|Q|}\int_Q |f(y) - m_Q f|\ dy$$

Remark 6.3. Note that $M^\# f \leqslant 2Mf$, so $M^\# f$ obeys all the corresponding bounds of Mf.

Lemma 6.4. *Let $f \in L^p(\mathbb{R}^d)$ for some $1 \leqslant p < \infty$. Then we have for all $\lambda > 0$ and $\gamma > 0$ that*

$$|\{M_d f > 2\lambda,\ M^\# f \leqslant \gamma\lambda\}| \leqslant 2^d\gamma|\{M_d f > \lambda\}| \tag{6.5}$$

Proof. The proof utilizes a Calderon-Zygmund type decomposition. We write

$$\{M_d f > \lambda\} = \bigcup_j Q_j$$

as a union of maximal dyadic cubes, so we just need to show that for any $Q \in \{Q_1, ..., Q_n, ...\}$, we have

$$|Q \cap \{M_d f > 2\lambda,\ M^{\#} f \leqslant \gamma\lambda\}| \leqslant 2^d \gamma |Q| \tag{6.6}$$

By maximality of Q we know that the parent cube $\tilde{Q}$ of Q satisfies

$$\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |f| \leqslant \lambda$$

and we notice that for x in the left hand set of (6.6), we have

$$M_d(f1_Q)(x) > 2\lambda$$

Hence

$$M_d((f - m_{\tilde{Q}} f)1_Q)(x) \geqslant M_d(f1_Q)(x) - m_{\tilde{Q}} f > \lambda$$

Therefore, we have

$$\begin{aligned}
|Q \cap \{M_d f > 2\lambda,\ M^{\#} f \leqslant \gamma\lambda\}| &\leqslant |\{M_d((f - m_{\tilde{Q}} f)1_Q) > \lambda\}| \\
&\leqslant \frac{1}{\lambda}\int_Q |f - m_{\tilde{Q}} f| \\
&= \frac{2^n|Q|}{\lambda}\frac{1}{|\tilde{Q}|}\int_{\tilde{Q}} |f - m_{\tilde{Q}} f| \\
&\leqslant \frac{2^n|Q|}{\lambda}\inf_{x\in Q} M^{\#} f(x) \\
&\leqslant 2^n \gamma |Q|
\end{aligned}$$

as desired. □

Lemma 6.7. *Suppose* $1 \leqslant p_0 < \infty$*, and* $f \in L^{p_0}$*. Then for all* $p_0 < p < \infty$*, we have some constant* C_p *such that*

$$\|M_d f\|_{L^p} \leqslant C_p \|M^{\#} f\|_{L^p}$$

Proof. We first truncate the layer-cake decomposition of $\|M_d f\|_{L^p}^p$ and denote

$$I_N := \int_0^N p\lambda^{p-1}|\{M_d f > \lambda\}|\ d\lambda$$

Then we have

$$\begin{aligned}
I_N &:= 2^p \int_0^{\frac{N}{2}} p\lambda^{p-1}|\{M_d f > 2\lambda\}| \\
&\leqslant 2^p\left(\int_0^{\frac{N}{2}} p\lambda^{p-1}|\{M_d f > 2\lambda,\ M^{\#} f \leqslant \gamma\lambda\}|\ d\lambda + \int_0^{\frac{N}{2}} p\lambda^{p-1}|\{M^{\#} > \gamma\lambda\}|\ d\lambda\right) \\
&\leqslant 2^{p+n}\gamma I_N + 2^p\gamma^{-p}\|M^{\#} f\|_{L^p}^p
\end{aligned}$$

Note that for every N, by Chebyshev inequality and L^{p_0} boundedness of M_d,

$$I_N \lesssim \int_0^N p\lambda^{p-p_0-1}\|f\|_{L^{p_0}}^{p_0} < \infty$$

so we have

$$I_N \leqslant \frac{2^p}{\gamma^p(1-2^{p+n}\gamma)}\|M^{\#}f\|_{L^p}^p$$

Choosing $\gamma > 0$ sufficiently small and sending $N \to \infty$, we get

$$\|M_d f\|_{L^p} \leqslant C_p\|M^{\#}f\|_{L^p}$$

as desired. □

Theorem 6.8 (Interpolation). *Let $1 \leqslant p_0 < \infty$. If T sublinear is bounded $L^{p_0} \to L^{p_0}$ and $L_c^\infty \to \mathrm{BMO}$, then for any $p_0 < p < \infty$, T is bounded $L^p \cap L^{p_0} \to L^p$, and T can be extended continuously to $L^p \to L^p$.*

Proof. Using a common density argument, we may assume f is smooth compactly supported. Remark 6.3 together with assumptions on T give us

$$\|(M^{\#} \circ T)f\|_{L^{p_0}} \leqslant C\|Tf\|_{L^{p_0}} \leqslant C'\|f\|_{L^{p_0}}$$

and also

$$\|(M^{\#} \circ T)f\|_{L^\infty} \leqslant \|Tf\|_* \leqslant \|f\|_{L^\infty}$$

By Marcinkiewicz interpolation theorem, $M^{\#} \circ T$ is bounded on L^p for all $p_0 \leqslant p \leqslant \infty$. Now, we note that

$$Tf(x) \leqslant (M_d \circ T)f(x) \quad \text{a.e.}$$

and Lemma 6.7 gives us

$$\|M_d(Tf)\|_{L^p} \leqslant C_p\|M^{\#}(Tf)\|_{L^p}$$

for $p_0 \leqslant p < \infty$. Thus

$$\|Tf\|_{L^p} \leqslant \|M_d(Tf)\|_{L^p} \leqslant C_p\|M^{\#}(Tf)\|_{L^p} \leqslant C_p'\|f\|_{L^p}$$

as desired. □

Remark 6.9. One can prove that if T is bounded $H^1 \to L^1$ and $L_c^\infty \to BMO$, then T is bounded on L^p for every $1 < p < \infty$. A proof can be found at the end of Chapter 3 of [8]. It is in fact essentially to the proof above.

7. Calderon-Zygmund Operators

This section concerns operators of the form

$$Tf(x) := \int K(x,y)f(y)\,dy, \quad x \in \mathbb{R}^n$$

where K is often assumed to be singular on the diagonal $\Delta := \{(x,x) : x \in \mathbb{R}^n\}$. The prototypical example is the Hilbert transform, where $K(x,y) = \frac{1}{x-y}$, and formally we have

$$Hf(x) = \int \frac{1}{x-y}f(y)\,dy$$

Of course, some extra work is needed if we want to make everything rigorous.

Definition 7.1 (Standard Kernel)**.** A standard kernel is a continuous function $K : \Delta^c \to \mathbb{C}$ for which there exists a constant $C > 0$ such that

$$|K(x,y)| \leqslant \frac{C}{|x-y|^n} \tag{7.2}$$

$$|\nabla_x K(x,y)| + |\nabla_y K(x,y)| \leqslant \frac{C}{|x-y|^{n+1}} \tag{7.3}$$

The smallest constant such that (7.2) and (7.3) hold is called the constant of the kernel K, and is denoted $C(K)$.

Proposition 7.4. *A standard kernel K satisfies the following properties:*

(1) For a cube $Q \subset \mathbb{R}^n$ centered at x, $f \in L^1_{loc}$,

$$\int_{\overline{Q}\setminus Q} |K(x,y)||f(y)| \; dy \leqslant C(Mf)(x) \tag{7.5}$$

(2) For $Q \subset \mathbb{R}^n$ and $x, x_0 \in Q$,

$$\int_{\overline{Q}^c} |K(x,y) - K(x_0,y)||f(y)| \; dy \leqslant C(Mf)(x_0) \tag{7.6}$$

(3) For any $x \in \mathbb{R}^n$ and $y_0 \in Q$,

$$\int_{\overline{Q}^c} \int_Q |K(x,y) - K(x,y_0)||f(y)| \; dydx \leqslant C|Q|(Mf)(y_0) \tag{7.7}$$

Definition 7.8. An operator T taking $C_c^\infty(\mathbb{R}^n)$ to $L^2(\mathbb{R}^n)$ is a Calderón-Zygmund operator (CZO) if:

(1) T extends to a bounded operator $L^2 \to L^2$.
(2) There exists a standard kernel K such that for every $f \in L_c^\infty(\mathbb{R}^n)$,

$$Tf(x) = \int K(x,y)f(y) \; dy \quad \text{a.e.}$$

on $(\text{Supp } f)^c$.

If T satisfies all properties above except for (1), then we say that T is **associated with a standard kernel** K.

If T is a CZO, then we denote

$$\|T\|_{CZ} := \|T\|_{L^2 \to L^2} + C(K) \tag{7.9}$$

We note that if (1) holds, then by a density argument we only needs to check (2) for a dense subclass of L_c^∞, e.g. $C_c^\infty(\mathbb{R}^n)$.

The following property of a Calderón-Zygmund Operator will be useful. Let's call it (H): For $Q \subset \mathbb{R}^n$, $a \in L^\infty(\mathbb{R}^n)$ such that Supp $a \subset Q$, we have

$$\int_{\overline{Q}} |Ta| \; dx \leqslant C\|a\|_{L^\infty}|Q| \tag{7.10}$$

The property (H) follows from L^2 boundedness of T and Hölder's inequality, since

$$\begin{aligned}\int_{\overline{Q}} |Ta| \, dx &\leqslant |\overline{Q}|^{\frac{1}{2}} \left(\int_{\overline{Q}} |Ta|^2 \, dx \right)^{\frac{1}{2}} \\ &\leqslant C|\overline{Q}|^{\frac{1}{2}} \left(\int_{\overline{Q}} |a|^2 \, dx \right)^{\frac{1}{2}} \\ &\leqslant C|Q| \|a\|_{L^\infty}\end{aligned}$$

We now state our main theorem on Calderón-Zygmund operators.

Theorem 7.11. *Let T be associated with a standard kernel. The following are equivalent:*

(1) T satisfies (H).
(2) T is bounded $H^{1,\infty}$ to L^1.
(3) T is bounded L_c^∞ to BMO.

Partial Proof. A complete proof can be found in Chapter 4 of [8]. Here we provide a proof of how (1) implies (3). Suppose $a \in L_c^\infty$. Let $Q \subset \mathbb{R}^d$ be a cube, and x_0 be the center of Q. Then we decompose

$$a = a_1 + a_2 := a \cdot 1_{\overline{Q}} + a \cdot 1_{\overline{Q}^c}$$

First we use (H) to obtain

$$\frac{1}{|Q|} \int_Q |Ta_1| \leqslant \frac{1}{|Q|} \int_{\overline{Q}} |Ta_1| \leqslant C\|a\|_{L^\infty}$$

Since Q is arbitrary and C is independent of Q, we have

$$\|a_1\|_* := \sup_Q \inf_\beta \frac{1}{|Q|} \int_Q |a - \beta| \leqslant C\|a\|_{L^\infty}$$

Now we study a_2. By Proposition 7.4, we have

$$\begin{aligned}\frac{1}{|Q|} \int_Q |Ta_2 - Ta_2(x_0)| &\leqslant \frac{1}{|Q|} \int_Q \int_{\overline{Q}^c} |K(x,y) - K(x,x_0)||a(y)| \, dydx \\ &\leqslant CMa(x_0) \\ &\leqslant C\|a\|_\infty\end{aligned}$$

showing that a_2 is in BMO if we reason as above. Hence $a \in BMO$, as desired. □

Remark 7.12. The proof above actually shows that T is bounded $L^\infty \to BMO$. Of course, one has to make sense of how to define T on L^∞ functions that don't necessarily vanish at infinity. Details can be found in [2].

The above theorem 7.11 together with the interpolation theorem 6.8 give the following powerful implication about Calderón-Zygmund operators. Note that if we use the interpolation mentioned in Remark 6.9 , we immediately get the result below.

Corollary 7.13. *Suppose T is a Calderón-Zygmund operator. Then T is bounded on L^p for $1 < p < \infty$.*

Proof. By Theorem 6.8 and Theorem 7.11, we have T bounded on L^p for $2 \leqslant p < \infty$. Now suppose $1 < q \leqslant 2$ and q' be defined such that $\frac{1}{p} + \frac{1}{p'} = 1$. For $f \in L^q$, note that

$$\|Tf\|_{L^q} = \inf_{\|g\|_{L^{q'}}=1} \int g(x)Tf(x)\,dx = \inf_{\|g\|_{L^{q'}}=1} \int T^*g(x)f(x)\,dx \tag{7.14}$$

The adjoint T^* is also a Calderon-Zygmund operator, associated with standard kernel

$$K^*(x,y) = K(y,x)$$

so Hölder's inequality gives

$$(7.14) \leqslant C_{q'}\|g\|_{L^{q'}}\|f\|_{L^q} \leqslant C_q\|f\|_{L^q}$$

as desired. □

Remark 7.15. All the work above, in particular, proves that the Hilbert transform is bounded on L^p for $1 < p < \infty$. One can also prove boundedness of Hilbert transform on L^p for $1 < p \leqslant 2$ using Marcinkiewicz interpolation theorem, and then get to $2 < p < \infty$ using a duality argument.

8. Littlewood-Paley Theory

We begin by constructing a dyadic partition of unity. Let $\varphi : \mathbb{R}^d \to [0,1]$ be such that $\varphi \in C_c^\infty$ and

$$\varphi(x) = \begin{cases} 1 & |x| \leqslant 1.4 \\ 0 & |x| > 1.42 \end{cases}$$

and we define $\psi(x) = \varphi(x) - \varphi(2x)$. For $N \in 2^{\mathbb{Z}}$, we define $\psi_N(x) = \psi(x/N)$. Then we notice that

$$\sum_{N\in 2^{\mathbb{Z}}} \psi_N(x) = 1$$

for all $x \neq 0$.

Definition 8.1. Let $f \in \mathcal{S}(\mathbb{R}^n)$. The Littlewood-Paley projection to frequencies $|\xi| \sim N$ is given by

$$\widehat{P_N f}(\xi) = \hat{f}(\xi)\psi_N(\xi)$$

or equivalently

$$P_N f = f * [N^d\check{\psi}(N\cdot)]$$

We also define

$$\widehat{P_{\leqslant N} f}(\xi) = \hat{f}(\xi)\varphi(\xi/N)$$

or equivalently

$$P_N f = f * [N^d\check{\varphi}(N\cdot)]$$

Moreover, we define

$$P_{>N} = 1 - P_{\leqslant N}$$

and define

$$P_{M\leqslant\ldots\leqslant N} = \sum_{M\leqslant K\leqslant N} P_K$$

Remark 8.2. The name "projection" can be slightly misleading, since actually $P_N^2 \neq P_N$.

Theorem 8.3 (Mikhlin Multiplier Theorem)**.** *Let $m : \mathbb{R}^d \backslash \{0\} \to \mathbb{C}$ be such that $|\nabla^k m(\xi)| \leqslant |\xi|^{-k}$ uniformly for $|\xi| \neq 0$ and $0 \leqslant k \leqslant d+2$. Then*

$$f \mapsto m(D)f$$

is bounded on L^p for $1 < p < \infty$.

The proof is basically use Littlewood-Paley decomposition to write $m = \sum_j m_j$, where each m_j is easy to handle. Using Calderón-Zygmund theory, we can show L^p boundedness of $m_j(D)$ with nice enough operator norm, and then we can sum in j to get L^p boundedness of m.

Theorem 8.4 (Bernstein Inequalities)**.** *Let $s \geqslant 0$ and $1 \leqslant p \leqslant q \leqslant \infty$. For simplicity of notation, we write L^p for $L^p(\mathbb{R}^d)$. We have the following inequalities:*

(1) $\|P_{\leqslant N} f\|_{L^p} \lesssim_{p,s,d} N^{-s} \||\nabla|^s P_{\geqslant N} f\|_{L^p}$.
(2) $\|P_{\leqslant N} |\nabla|^s f\|_{L^p} \lesssim_{p,s,d} N^s \|P_{\leqslant N} f\|_{L^p}$.
(3) $\|P_N |\nabla|^{\pm s} f\|_{L^p} \sim_{p,s,d} N^{\pm s} \|P_N f\|_{L^p}$.
(4) $\|P_{\leqslant N} f\|_{L^q} \lesssim N^{\frac{d}{p} - \frac{d}{q}} \|P_{\leqslant N} f\|_{L^p}$.
(5) $\|P_N f\|_{L^q} \lesssim N^{\frac{d}{p} - \frac{d}{q}} \|P_N f\|_{L^p}$

We also have a characterization of the Sobolev norms:

Proposition 8.5. *For $f \in \mathcal{S}$,*

(1) $\|f\|_{\dot{H}^s} \sim \left(\sum_{N \in 2^{\mathbb{Z}}} N^{2s} \|P_N f\|_{L^2}^2\right)^{1/2} = \left\| N^s \|P_N f\|_{L^2} \right\|_{l^2_N}$
(2) $\|f\|_{H^s} \sim \|P_{\leqslant 1} f\|_{L^2} + \left(\sum_{N > 1} N^{2s} \|P_N f\|_{L^2}^2\right)^{1/2}$

Proposition 8.6 (Product Estimate)**.** *If $f, g \in \mathcal{S}(\mathbb{R}^d)$ and $a, b \geqslant 0$, then the following holds:*

$$\||\nabla|^a f |\nabla|^b g\|_{L^2(\mathbb{R}^d)} \lesssim \|f\|_{H^{a+b}(\mathbb{R}^d)} \|g\|_{L^\infty} + \|f\|_{L^\infty} \|g\|_{H^{a+b}(\mathbb{R}^d)}$$

Proof. The proof of these product estimates are all of the same spirit, so what we are doing here may generalize to other product-type estimates. We decompose

$$\begin{aligned} |\nabla|^a f |\nabla|^b g &= \left(\sum_N P_N |\nabla|^a f\right) \left(\sum_M P_M |\nabla|^b g\right) \\ &= \sum_{N,M} (P_N |\nabla|^a f)(P_M |\nabla|^b g) \\ &= \sum_M (P_{\leqslant M} |\nabla|^a f)(P_M |\nabla|^b g) + (P_{>M} |\nabla|^a f)(P_M |\nabla|^b g) \end{aligned}$$

We shall only show that

$$\left\| \sum_M (P_{\leqslant M} |\nabla|^a f)(P_M |\nabla|^b g) \right\|_{L^2} \lesssim \|f\|_{L^\infty} \|g\|_{H^{a+b}}$$

since the proof that

$$\left\| \sum_M (P_{>M} |\nabla|^a f)(P_M |\nabla|^b g) \right\|_{L^2} \lesssim \|g\|_{L^\infty} \|f\|_{H^{a+b}}$$

is analogous.

We want to look at

$$
\begin{aligned}
&P_K\big((P_{\leqslant M}|\nabla|^a f)(P_M|\nabla|^b g)\big)\\
&= P_K\big((P_{\leqslant M/8}|\nabla|^a f)(P_M|\nabla|^b g)\big) + P_K\big((P_{8/M\leqslant \cdot \leqslant M}|\nabla|^a f)(P_M|\nabla|^b g)\big)\\
&=: A + B
\end{aligned}
\tag{8.7}
$$

Note that the Fourier support of $(P_{\leqslant M/8}|\nabla|^a f)(P_M|\nabla|^b g)$ is on within the annulus $\{M/16 \leqslant |\xi| \leqslant 4M\}$, so A is non-zero only when $M/32 \leqslant K \leqslant 8M$. Moreover, we can see that B is also non-zero only when $K \sim M$. Hence

$$
\begin{aligned}
&P_K\bigg(\sum_M (P_{\leqslant M}|\nabla|^a f)(P_M|\nabla|^b g)\bigg)\\
&= \sum_{M\sim K} P_K\big((P_{\leqslant M/8}|\nabla|^a f)(P_M|\nabla|^b g)\big) + P_K\big((P_{8/M\leqslant \cdot \leqslant M}|\nabla|^a f)(P_M|\nabla|^b g)\big)
\end{aligned}
$$

and thus

$$
\begin{aligned}
&\bigg\|P_K\bigg(\sum_M (P_{\leqslant M}|\nabla|^a f)(P_M|\nabla|^b g)\bigg)\bigg\|_{L^2}\\
&\leqslant \sum_{M\sim K} \|P_K\big((P_{\leqslant M/8}|\nabla|^a f)(P_M|\nabla|^b g)\big)\|_{L^2} + \|P_K\big((P_{8/M\leqslant \cdot \leqslant M}|\nabla|^a f)(P_M|\nabla|^b g)\big)\|_{L^2}
\end{aligned}
$$

We define $\varphi(\frac{\cdot}{M})$ and $\psi(\frac{\cdot}{M})$ to be the multiplier functions used in $P_{\leqslant M}$ and P_M, and also

$$
\varphi_a(\xi) := |\xi|^a \varphi(\xi), \quad \psi_b(\xi) := |\xi|^b \psi(\xi)
$$

$$
\varphi_{a,M}(\xi) := \varphi_a(\xi/M), \quad \psi_{b,M}(\xi) := \psi_b(\xi/M)
$$

so that

$$
\begin{aligned}
(P_{\leqslant M}|\nabla|^a f)(P_M|\nabla|^b g) &= \mathscr{F}^{-1}\bigg(M^{a+b}(\varphi_{a,M}\cdot \widehat{f})(\psi_{b,M}\cdot \widehat{g})\bigg)\\
&= M^{a+b}(\varphi_{a,M}^\vee * f)(\psi_{b,M}^\vee * g)\\
&= (\varphi_{a,M}^\vee * f)(M^{a+b}\psi_{b,M}^\vee * g)
\end{aligned}
$$

Then by Young's convolution inequality and the fact that $\psi_{b,M}$ is supported on $\{|\xi| \sim M\}$, we have

$$
\begin{aligned}
\big\|P_K\big((P_{\leqslant M}|\nabla|^a f)(P_M|\nabla|^b g)\big)\big\|_{L^2} \big\|(P_{\leqslant M}|\nabla|^a f)(P_M|\nabla|^b g)\big\|_{L^2}
&\lesssim \|f\|_{L^\infty} M^{a+b} \|\psi_{b,M}^\vee * g\|_{L^2}\\
&\lesssim M^{a+b}\|f\|_{L^\infty}\|P_M g\|_{L^2}
\end{aligned}
$$

and similar argument gives

$$
\|P_K\big((P_{8/M\leqslant \cdot \leqslant M}|\nabla|^a f)(P_M|\nabla|^b g)\big)\|_{L^2} \lesssim M^{a+b}\|f\|_{L^\infty}\|P_M g\|_{L^2}
$$

Therefore,

$$\begin{aligned}\Big\|\sum_M (P_{\leqslant M}|\nabla|^a f)(P_M|\nabla|^b g)\Big\|_{L^2}^2 &= \sum_K \Big\|P_K\Big(\sum_M (P_{\leqslant M}|\nabla|^a f)(P_M|\nabla|^b g)\Big)\Big\|_{L^2}^2 \\ &\lesssim \|f\|_{L^\infty}\sum_K\sum_{M\sim K} M^{2a+2b}\|P_M g\|_{L^2}^2 \\ &\lesssim \|f\|_{L^\infty}\sum_M M^{2a+2b}\|P_M g\|_{L^2}^2 \\ &\lesssim \|f\|_{L^\infty}\|g\|_{H^{a+b}}\end{aligned}$$

as desired. □

9. Pseudodifferential Operators

Pseudodifferential operators refer to operators of the form

$$T_a f(x) := \int a(x,\xi)\widehat{f}(\xi)e^{2\pi i x\xi}\,d\xi$$

where a is called the **symbol** of the operator.

Example 9.1. (1) The simplest case is when $a \equiv 1$. In this case T_a is the inverse Fourier transform of $\widehat{f}$, meaning that formally it is just the identity.
(2) Suppose $a = x$. Then formally $T_a f(x) = xf(x)$.
(3) Suppose $a = \xi$. Then since multiplying in the frequency space corresponds to differentiating in the physical space, $T_a f(x) \sim f'(x)$.

Definition 9.2. We say T_a is a pseudodifferential operator of order m if

$$|\partial_x^\beta \partial_\xi^\alpha a(x,\xi)| \leqslant C_{\alpha,\beta}(1+|\xi|)^{m-|\alpha|}$$

for any $\alpha, \beta \in \mathbb{N}$. We denote the collection of such operators by S^m.

We first present the following useful lemma, which exemplifies what is called the TT^* argument.

Lemma 9.3 (Cotlar-Stein Lemma)**.** *Let H be a Hilbert space and $\{T_j\}_{j\in\mathbb{Z}}$ a sequence of bounded operators on H with adjoints $\{T_j^*\}_{j\in\mathbb{Z}}$. Let $\{a(j)\}_{j\in\mathbb{Z}}$ be a sequence of non-negative numbers such that*

$$\|T_iT_j^*\| + \|T_i^*T_j\| \leqslant a(i-j)$$

Then for all integers n and m, $n \leqslant m$, we have

$$\Big\|\sum_{i=n}^m T_i\Big\| \leqslant \sum_{-\infty}^{\infty} a(i)^{1/2}$$

Proof. First of all let $S := \sum_n^m T_i$. Then S is a bounded operator on H. By the TT^* theorem, we have

$$\|S\| = \|(SS^*)^k\|^{1/2k} \tag{9.4}$$

Meanwhile, we expand to obtain

$$(SS^*)^k = \sum_{n\leqslant j_1,\ldots,j_{2k}\leqslant m} T_{j_1}T_{j_2}^*\ldots T_{2k-1}T_{2k}^*$$

On the one hand we have

$$\|(SS^*)^k\| \leqslant \|T_{j_1}T_{j_2}^*\|...\|T_{2k-1}T_{2k}^*\| \leqslant a(j_1-j_2)...a(j_{2k-1}-j_{2k})$$

and on the other hand we have

$$\|(SS^*)^k\| \leqslant \|T_{j_1}\|\|T_{j_2}^*T_{j_3}\|...\|T_{2k-2}^*T_{2k-1}\|\|T_{2k}^*\| \leqslant a(0)^{1/2}a(j_2-j_3)...a(j_{2k-2}-j_{2k-1})a(0)^{1/2}$$

So taking geometric mean of the right hand sides above we get

$$\begin{aligned}\|(SS^*)^k\| &\leqslant a(0)^{1/2}\sum_{n\leqslant j_1,\dots,j_{2k}\leqslant m} a(j_1-j_2)^{1/2}a(j_2-j_3)^{1/2}...a(j_{2k-1}-j_{2k})^{1/2}\\ &\leqslant a(0)^{1/2}(m-n+1)\left(\sum_{-\infty}^{\infty}a(i)^{1/2}\right)^{2k-1}\end{aligned}$$

so by (11.3) we have

$$\|S\| \leqslant a(0)^{1/4k}(m-n+1)^{1/2k}\left(\sum_{-\infty}^{\infty}a(i)^{1/2}\right)^{\frac{2k-1}{2k}}$$

and sending $k\to\infty$ we get

$$\|S\| = \left\|\sum_{i=n}^{m}T_i\right\| \leqslant \sum_{-\infty}^{\infty}a(i)^{1/2}$$

as desired. □

We have the following application of the Cotlar's lemma to Hilbert transform:

Example 9.5. For $f\in L^2(\mathbb{R})$ we define the truncated Hilbert transform

$$T_jf(x) = \int_{2^j\leqslant|t|\leqslant 2^{j+1}}\frac{f(x-t)}{t}\,dt$$

We note that

$$\begin{aligned}|T_jf(x)| &\leqslant \int_{2^j\leqslant|t|\leqslant 2^{j+1}}\left|\frac{f(x-t)}{t}\right|dt\\ &\leqslant \frac{4}{2^{j+2}}\int_{|t|\leqslant 2^{j+1}}|f(x-t)|\,dt\\ &\leqslant 4Mf(x)\end{aligned}$$

where M is the Hardy-Littlewood maximal operator. Thus T_j is bounded on L^2. Using Cotlar's lemma, we can show that any finite sum of the T_j's are uniformly bounded.

Theorem 9.6 (Calderón-Vaillancourt)**.** *Let T_a be a pseudodifferential operator of order* 0*. Then T_a extends to a bounded operator on L^2.*

A proof can be found on [6].

10. T(1) Theorem

10.1. **Carleson's Measure.** The $T(1)$ theorem was originally published in [1]. Here we follow the proof of $T(1)$ Theorem in [5], for which some preliminaries on Carleson's measures are needed.

Definition 10.1 (Carleson's Measure)**.** A positive measure ν on $\mathbb{R}^{n+1}_+$ is a Carleson's measure if for every cube $Q \subset \mathbb{R}^n$, we have

$$\nu(Q \times (0, l(Q))) \leqslant C|Q|$$

where $l(Q)$ is the side length of Q. The infimum over all possible values of C is called the Carleson's constant and is usually denoted $\|\nu\|$.

Given an open subset $E \subset \mathbb{R}^n$, we let

$$\widehat{E} := \{(x, t) \in \mathbb{R}^{n+1}_+ : B(x, t) \subset E\}$$

Then we have the following lemma.

Lemma 10.2. *If ν is a Carleson's measure in $\mathbb{R}^{n+1}_+$ and $E \subset \mathbb{R}^n$ is open, then*

$$\nu(\widehat{E}) \leqslant C\|\nu\||E|$$

Proof. The proof is done using Calderon-Zygmund decomposition. □

Theorem 10.3. *Let ϕ be a bounded, integrable function which is positive, radial, and decreasing. For $t > 0$, let $\phi_t(x) = t^{-n}\phi(t^{-1}x)$. Then a measure ν is a Carleson's measure if and only if for every p, $1 < p < \infty$,*

$$\int_{\mathbb{R}^{n+1}_+} |\phi_t * f(x)|^p \, d\nu(x, t) \leqslant C \int_{\mathbb{R}^n} |f(x)|^p \, dx$$

The constant C is comparable to $\|\nu\|$.

Here is another important theorem that connects Carleson's measure to BMO functions.

Theorem 10.4. *Let $b \in BMO$ and $\psi \in \mathcal{S}(\mathbb{R}^n)$ be such that $\int \psi = 0$. Then the measure ν defined by*

$$d\nu = |b * \psi_t(x)| \frac{dxdt}{t}$$

is a Carleson's measure such that $\|\nu\| \lesssim \|b\|_*^2$.

Proof. □

Corollary 10.5. *Let ϕ and ψ be as in the previous two theorems, and $b \in BMO$. Then for $1 < p < \infty$ we have*

$$\int_{\mathbb{R}^{n+1}_+} |\phi_t * f(x)|^p \cdot |(b * \psi_t)(x)| \frac{dxdt}{t} \leqslant C \int_{\mathbb{R}^n} |f(x)|^p \, dx$$

Proof. The proof is essentially the previous two theorems put together. □

10.2. **T(1) Theorem and Applications.** In order to show that an operator is a Calderón-Zygmund operator, a key assumption to verify is L^2 boundedness. The $T(1)$ theorem provides a criterion, saying that as long as the operator and its adjoint sends the constant function 1 to BMO and satisfies some other reasonable assumptions, then it is actually bounded on L^2. Of course, some work is needed to make everything rigorous.

To simplify the presentation, we state everything in the 1-dimensional domain $\mathbb{R}$.

Definition 10.6 (Weak Boundedness Property, as in [5])**.** Given a function f, we define

$$f_t^x(y) := \frac{1}{t} f\left(\frac{y-x}{t}\right)$$

We say that T sastisfies the weak boundedness property (WBP), if there exists $\phi \in C_c^\infty(\mathbb{R})$ radially symmetric and its derivative $\psi = \phi'$, such that for some $\varepsilon > 0$,

$$|\langle T\psi_t^v, \phi_t^u\rangle| \lesssim \frac{1}{t}\frac{1}{1+|\frac{u-v}{t}|^{1+\varepsilon}}, \quad |\langle T^*\psi_t^v, \phi_t^u\rangle| \lesssim \frac{1}{t}\frac{1}{1+|\frac{u-v}{t}|^{1+\varepsilon}} \quad \text{for every } t>0. \tag{10.7}$$

For convenience, we fix the notation

$$p_t(x) := \frac{1}{t}\frac{1}{1+|\frac{x}{t}|^{1+\varepsilon}}, \ p := p_1$$

Theorem 10.8. *Let $T : \mathcal{S} \to \mathcal{S}'$ be a linear operator. If*

(1) T sastisfies the weak boundedness property (WBP);
(2) $T(1) \in$ BMO;
(3) $T^*(1) \in$ BMO;

Then T is bounded $L^2 \to L^2$.

We have several remarks on the $T(1)$ Theorem before we start the proof. First of all we need to make clear what does $T(1) \in BMO$ mean. One way to formulate it is the following: we say $T(1) \in BMO$ if there exists a BMO function b such that

$$\langle T(1), g\rangle = \langle b, g\rangle$$

for all $g \in C_c^\infty$ with mean 0.

The next remark is given by the following proposition.

Proposition 10.9. *Suppose T is a singular integral operator associated with an anti-symmetric kernel K, i.e.*

$$Tf(x) = p.v. \int_{|x-y|>\varepsilon} K(x,y)f(y)\ dy = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} K(x,y)f(y)\ dy$$

for some $K(x,y) = -K(y,x)$ and $f \in C_c^\infty(\mathbb{R})$, then T automatically satisfies WBP. Moreover, T is bounded on L^2 if and only if $T(1) \in BMO$.

Proof. Let $t > 0$. Suppose $|x-y| > 3t$. Then since ψ_t^x has mean 0, we can write

$$\begin{aligned}
\langle T\psi_t^x, \phi_t^y\rangle &= \iint (K(z,u) - K(z,x))\phi_t^y(z)\psi_t^x(u)\ dudz \\
&\lesssim \iint \frac{1}{|x-y|^2}\cdot|\phi_t^y(z)|\cdot|\psi_t^x(u)|\ dudz \\
&\lesssim \frac{t}{|x-y|^2}
\end{aligned}$$

where the first inequality follows from standard estimates and triangular inequality. Note that

$$\frac{t}{|x-y|^2} = \frac{1}{t}\frac{1}{|\frac{x-y}{t}|^2} \lesssim \frac{1}{t}\frac{1}{1+|\frac{x-y}{t}|^2}$$

where we have the last inequality because $|\frac{x-y}{t}| > 3$.

Now suppose $|x-y| \leqslant 3t$. Using antisymmetry, we may write

$$\langle T\psi_t^x, \phi_t^y\rangle = p.v. \iint K(z,u)(\phi_t^y(z)\psi_t^x(u) - \phi_t^y(u)\psi_t^x(z))\ dudz$$

By mean value theorem, it is easy to verify that

(10.10)
$$\begin{aligned}|\phi_t^y(z)\psi_t^x(u) - \phi_t^y(u)\psi_t^x(z)| &\leqslant |\phi_t^y(z)\psi_t^x(u) - \phi_t^y(u)\psi_t^x(u)| + |\phi_t^y(u)\psi_t^x(u) - \phi_t^y(u)\psi_t^x(z)| \\ &\leqslant (\|\psi_t^x\|_{L^\infty}\|(\phi_t^y)'\|_{L^\infty} + \|(\psi_t^x)'\|_{L^\infty}\|\phi_t^y\|_{L^\infty})|u-z| \\ &\lesssim \frac{|u-z|}{t^3}\end{aligned}$$

Then we can use standard estimates to get

$$\langle T\psi_t^x, \phi_t^y\rangle \lesssim \int_A \frac{1}{|u-z|}\frac{|u-z|}{t^3}\ dudz$$

where

$$A := \{(u,z) : |\phi_t^y(z)\psi_t^x(u) - \phi_t^y(u)\psi_t^x(z)| > 0\}$$

Since ϕ is compactly supported, we have $|u-x| \lesssim t$, $|z-y| \lesssim t$. Since $|x-y| < 3t$, we know that $|u-z| \lesssim t$. It follows that $|A| \lesssim t^2$, and thus

$$\langle T\psi_t^x, \phi_t^y\rangle \lesssim \frac{1}{t}$$

Since $|x-y| \leqslant 3t$, note that

$$\frac{1}{t} \lesssim \frac{1}{t}\frac{1}{1+|\frac{x-y}{t}|^2}$$

So we get the desired result. □

Proof of T(1) Theorem. Let ϕ and ψ be as above. We denote $P_tf = \phi_t * f$ and $Q_tf = \psi_t * f$, and since P_t is an approximation of identity, for $f \in \mathcal{S}$, we write

$$Tf = \lim_{t\to 0} P_t^2TP_t^2f = \lim_{\varepsilon\to 0}\int_\varepsilon^{1/\varepsilon} \partial_t(P_t^2TP_t^2f)\ dt$$

To show that the first equality holds, we claim that if $T_n \to T$ in $\mathcal{S}'$ and $f_n \to f \in \mathcal{S}$, then $T_nf_n \to Tf$. This claim is a corollary of the following Banach-Steinhaus theorem in Frechét space.

Lemma 10.11 (Banach-Steinhaus). *Let X be a Frechét space and Y be a normed space, and let $\{\phi_\alpha\}_\alpha$ be a family of continuous linear maps $X \to Y$. If $\sup_\alpha \|\phi(x)\| < \infty$ for every $x \in X$, then $\{\phi_\alpha\}_\alpha$ is a equicontinuous family.*

For convenience, we abuse notation by writing

$$\int_0^\infty \partial_t(P_t^2 T P_t^2 f)\ dt = \lim_{\varepsilon\to 0}\int_\varepsilon^{1/\varepsilon} \partial_t(P_t^2 T P_t^2 f)\ dt$$

and using chain rule and linearity of T, we may write

$$\int_0^\infty \partial_t(P_t^2 T P_t^2 f)\ dt = \int_0^\infty (t\partial_t P_t^2)(TP_t^2 f)\ \frac{dt}{t} + \int_0^\infty P_t^2 T(t\partial_t^2 P_t^2 f)\ \frac{dt}{t}$$

The second integrand on the right hand side is the conjugate of the first, so we only study the first integral. Taking Fourier transform in space, we have

$$\begin{aligned}\widehat{t\partial_t P_t^2 g} &= 2t\phi(t\xi)\xi\widehat{\phi}'(t\xi)\widehat{g}(\xi)\\ &= 2\widehat{\psi}(t\xi)\widehat{\psi_1}(t\xi)\widehat{g}(\xi)\\ &= 2\widehat{Q_{1,t}Q_t}g\end{aligned}$$

where $\psi_1(x) = x\phi(x)$ and $Q_{1,t}g = \psi_{1,t} * g$. Thus the integral we are interested in can be rewritten as

$$-2\int_0^\infty Q_{1,t}Q_t TP_t^2 f\frac{dt}{t} =: -2\int_0^\infty Q_{1,t}L_t P_t f\frac{dt}{t}$$

so we have

$$\begin{aligned}L_t g = Q_t TP_t g &= \psi_t * [T(\psi_t * g)]\\ &= \int \psi_t(x-y)T(\psi_t * g)(y)\ dy\\ &= \iint T^*(\psi_t^x)(y)\phi_t^y(z)g(z)\ dzdy\\ &= \int \left(\int T^*(\psi_t^x)(y)\phi_t^z(y)\ dy\right)g(z)\ dz\\ &= \int \langle T^*(\psi_t^x), \phi_t^z\rangle g(z)\ dz\end{aligned}$$

For convenience, we define

$$l_t(x,z) := \langle T^*(\psi_t^x), \phi_t^z\rangle$$

Observe that the assumptions of T(1) theorem give us

$$l_t(x,z) \lesssim p_t(x-z)$$

and that

$$L_t(1) = \langle T^*(\psi_t^x), 1\rangle = \langle \psi_t^x, T(1)\rangle = \psi_t * b$$

Now, we decompose

$$\begin{aligned}\int_0^\infty Q_{1,t}L_t P_t f\frac{dt}{t} &= \int_0^\infty Q_{1,t}L_t(1)P_t f\frac{dt}{t} + \int_0^\infty Q_{1,t}(L_t P_t f - L_t(1)P_t f)\frac{dt}{t}\\ &=: \int_0^\infty Q_{1,t}\{(\psi_t * f)(\psi_t * b)\}\frac{dt}{t} + E(f)\end{aligned}$$

and we begin by estimating $E(f)$. By duality, assume $g \in \mathscr{S}$ satisfies $\|g\|_{L^2} = 1$, then

$$\begin{aligned}\langle g, E(f)\rangle &= \int\int_0^\infty Q_{1,t}(g)(L_tP_tf - L_t(1)P_tf)\frac{dtdx}{t} \\ &\leqslant \left(\int\int_0^\infty |Q_{1,t}(g)|^2\frac{dtdx}{t}\right)^{1/2}\left(\int\int_0^\infty \left|\int l_t(x,y)(P_f(y) - P_tf(x))dy\right|^2\frac{dtdx}{t}\right)^{1/2}\end{aligned}$$

By Plancherel,

$$\int\int_0^\infty |Q_{1,t}(g)|^2\frac{dtdx}{t} = \int_0^\infty \int |\widehat{\psi_1}(t\xi)\widehat{g}(\xi)|^2\frac{d\xi dt}{t}$$

With a change of variable $t = t\xi$, the above equation becomes

$$\int |\widehat{g}(\xi)|^2\left(\int |\widehat{\psi_1}(t)|^2\frac{dt}{t}\right)d\xi \lesssim \|g\|_{L^2}^2 = 1$$

At the mean time, Minkowski integral inequality gives

$$\begin{aligned}&\int\int_0^\infty \left|\int l_t(x,y)(P_tf(y) - P_tf(x))dy\right|^2\frac{dtdx}{t} \\ &\lesssim \int_0^\infty \iint |l_t(x,y)(P_tf(y) - P_tf(x))dy|^2\frac{dxdydt}{t} \\ &= \int_0^\infty \iint |p_t(u)|^2|P_tf(y) - P_tf(y+u)|^2\frac{dudydt}{t} \\ &= \int_0^\infty \iint |p_t(u)|^2|\widehat{\phi}(t\xi)|^2|e^{iu\xi} - 1|^2|\widehat{f}(\xi)|^2\frac{dud\xi dt}{t} \\ &\lesssim_\delta \left(\int |p(u)|^2u^\delta du\right)^{1/2}\left(\int_0^\infty |\widehat{\phi}(t)|^2t^{\delta-1}dt\right)^{1/2}\left(\int |\widehat{f}(\xi)|^2d\xi\right)^{1/2}\end{aligned}$$

If we fix a $\delta < \varepsilon$, the above integral is

$$\lesssim \|f\|_{L^2}^2$$

as desired. For the term

$$\int_0^\infty Q_{1,t}\{(\psi_t * f)(\psi_t * b)\}\frac{dt}{t}$$

we use a duality argument as above, and appeal to Corollary 10.5 to get that it is bounded by an absolute constant. Now the proof is complete. □

As an application to $T(1)$ theorem, we study certain Cauchy integral operators. Suppose we have a Lipshitz graph $\Gamma := \{(x, A(x)) : x \in \mathbb{R}\}$. For convenience, we denote $z(x) := x + iA(x)$, and consider the operator

$$Tf(x) := p.v.\int \frac{z'(y)}{z(x) - z(y)}f(y)\ dy$$

Proposition 10.12. *There is a sufficiently small constant $c > 0$ such that T is bounded on $L^2(\mathbb{R})$ as long as z is Lipschitz with $\|A'\|_{L^\infty} \leqslant c$.*

Proof. We define the operator

$$Cf(x) := p.v. \int \frac{f(y)}{z(x) - z(y)} dy = p.v. \int \frac{f(y)}{x - y + i(A(x) - A(y))} dy$$

and are done with the proof if we can show C is bounded on L^2. To see this, assume for now that C is bounded on L^2, then $Tf = C(fz')$ and thus

$$\|Tf\|_{L^2} = \|C(fz')\|_{L^2} \leqslant \|fz'\|_{L^2} \leqslant \|f\|_{L^2}$$

where we have the last inequality since $\|z'\|_{L^\infty}$ is bounded.

Now our task is reduced to showing boundedness of C on L^2, which we establish using the $T(1)$ theorem. Since the kernel of C is antisymmetric, we only need to show that $C(1) \in BMO$.

Direct calculation gives

$$Cf(x) = \sum_n (-i)^n p.v. \int \frac{f(y)}{x - y} \left(\frac{A(x) - A(y)}{x - y} \right)^n dy$$

assuming that $\|A'\|_{L^\infty}$ is small enough. We denote

$$C_n f(x) := \int \frac{f(y)}{x - y} \left(\frac{A(x) - A(y)}{x - y} \right)^n dy$$

and claim that $C_n(1) = C_{n-1}(A')$. Integrating by parts at least formally, we have

$$\begin{aligned}
C_n(1) &= \int \left(\frac{A(x) - A(y)}{x - y} \right)^n \frac{1}{x - y} \, dy \\
&= \frac{1}{n} \int (A(x) - A(y))^n \partial_y \frac{1}{(x - y)^n} \, dy \\
&= \int \left(\frac{A(x) - A(y)}{x - y} \right)^{n-1} \frac{A'(y)}{x - y} \, dy \\
&= C_{n-1}(A')
\end{aligned}$$

which establishes the claim. One just needs to be slightly more careful for a rigorous deduction, and we do not show all the details here.

We want to show that there exists some $C > 0$ such that

$$\|C_k(1)\|_* \leqslant C^{k+1} \|A'\|_{L^\infty}^{k-1} \tag{10.13}$$

uniformly in k, and we achieve it by induction. For the base case, note that

$$C_1(1) = C_0(A') = \pi H(A')$$

and thus

$$\|C_1(1)\|_* \leqslant C \|A'\|_{L^\infty}$$

in view of Theorem 7.11 and Remark 7.12. Now, suppose (10.13) is true for k. We have

$$\|C_{k+1}\|_{L^\infty \to BMO} \leqslant C_1(C_2(k+1)\|A'\|_{L^\infty}^k + \|C_k\|_{L^2 \to L^2})$$

if we look at the proof we had for Theorem 7.11 and keep track of the dependence on constants. Here $\|C_k\|_{L^2 \to L^2} < \infty$ by induction hypothesis, antisymmetry of the kernel of C_k, and the $T(1)$ theorem. Keeping track of dependence on constants in the proof of T(1) Theorem, we get

$$\|C_k\|_{L^2 \to L^2} \leqslant C_3 C_4^{k+1} \|A'\|_{L^\infty}^k$$

Choosing C sufficiently large with respect to C_1, C_2, C_3, C_4 and independent of k, we get

$$\|C_{k+1}\|_{L^\infty \to BMO} \leqslant C^{k+2} \|A'\|_{L^\infty}$$

and the conclusion follows from induction. Therefore,

$$\|C(1)\|_* \leqslant \sum_k \|C_k(1)\|_* \leqslant \sum_k C^{k+1} \|A'\|_{L^\infty}^{k-1}$$

which is finite if $\|A'\|_{L^\infty}$ is sufficiently small. Now the proof is complete. □

11. Cauchy Integral and Hilbert Transform

In this section, we explore the connection between Cauchy integral and Hilbert transform. This connection and things of similar spirit are used in the study of free boundary problems.

We assume $\eta : \mathbb{R} \to \mathbb{R}$ is a Lipschitz function such that

$$\lim_{x \to -\infty} \eta(x) = \lim_{x \to +\infty} \eta(x) = 0$$

and let

$$\Omega := \{(x, y) : x \in \mathbb{R}, \quad y < \eta(x)\}$$

and

$$\Sigma := \partial\Omega = \{(x, \eta(x)) : x \in \mathbb{R}\}$$

Suppose $z : \mathbb{R} \to \mathbb{C}$ is a parametrization Σ, we define the Hilbert transform associated with Σ to be

$$\mathfrak{H} f(\alpha) = \frac{1}{\pi i}\, p.v. \int \frac{f(\beta) z_\beta(\beta)}{z(\alpha) - z(\beta)}\, d\beta := \lim_{\varepsilon \to 0} \frac{1}{\pi i} \int_{|\beta - \alpha| > \varepsilon} \frac{f(\beta) z_\beta(\beta)}{z(\alpha) - z(\beta)}\, d\beta$$

The goal of this section is to prove the following proposition that characterizes holomorphic functions on $\Omega(t)$.

Proposition 11.1. *Let $g \in L^p$ for some $1 < p < \infty$. Then*

(1) g is a boundary value of a holomorphic function G such that $G(z) \to 0$ as $|z| \to \infty$ if and only if $\mathfrak{H} g = g$.
(2) $\frac{1}{2}(I + \mathfrak{H})g$ is the boundary value of a holomorphic function $\mathfrak{G}$ on Ω such that $\mathfrak{G}(z) \to 0$ as $|z| \to \infty$.
(3) $\mathfrak{H}1 = 0$.

Let $f : \Sigma \to \mathbb{C}$ be induced by g such that

$$f(z(\beta)) := g(\beta)$$

We also define

$$G(z) := \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{f(z(\beta)) z'(\beta)}{z - z(\beta)}\, d\beta = \frac{1}{2\pi i} \int_{\Sigma} \frac{f(\zeta)}{z - \zeta}\, d\zeta$$

on Ω.

Henceforth, we suppose z is an arclength parametrization of Σ such that there are constants $c, C > 0$ such that

$$c \leqslant \left| \frac{z(\alpha) - z(\beta)}{\alpha - \beta} \right| \leqslant C$$

for $\alpha, \beta \in \mathbb{R}$.

Now, we state and prove several preparatory lemmas before we begin the proof of the theorem.

Lemma 11.2 (Plemelj Formula)**.** *Let $g \in C_c^\infty(\mathbb{R})$. For any $\alpha \in \mathbb{R}$, we have*

$$\lim_{z \to z(\alpha)} G(z) = \frac{1}{2}g(\alpha) + \frac{1}{2}\mathfrak{H}g(\alpha) \tag{11.3}$$

where in the limit we have $z \in \Omega$.

Proof of Lemma. We begin with the case that $g \in C_c^\infty(\mathbb{R})$. Then there are some $\beta_1 < \beta_2 \in \mathbb{R}$ such that g is supported in (β_1, β_2). We define

$$\Gamma := \{z(\beta) : \beta_1 \leqslant \beta \leqslant \beta_2\}$$

so that

$$\frac{1}{2\pi i}\int_\Sigma \frac{f(\zeta)}{z-\zeta}\, d\zeta = \frac{1}{2\pi i}\int_\Gamma \frac{f(\zeta)}{z-\zeta}\, d\zeta$$

Now, we consider an extension of f such that $f(z)$ is constant on $\{\mathrm{Re}(z) = a\}$ for every $a \in \mathbb{R}$, and abusing notations a little bit we still write this extended function as $f(z)$. Note that $f(z)$ is Lipschitz. Then we can write

$$\int_\Gamma \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta + \int_\Gamma \frac{f(z)}{\zeta - z}\, d\zeta = \int_\Gamma \frac{f(\zeta) - f(z)}{\zeta - z}\, d\zeta + \log\left(\frac{\zeta_2 - z}{\zeta_1 - z}\right)$$

where the log function is defined on $\mathbb{C} - \{ia : a \geqslant 0\}$. Now, since f is Lipschitz, the first summand above behaves well if we send $z \to z(\alpha)$ by dominated convergence, and the second summand also behaves well if we send $z \to z(\alpha)$ by continuity of log on its domain.

Having shown that the limit on the left hand side of (11.3) exists, we compute that it equals the right hand side. In particular, it suffices to show

$$\lim_{\varepsilon \to 0} G(z(\alpha) - i\varepsilon z'(\alpha)) = \frac{1}{2}g(\alpha) + \frac{1}{2}\mathfrak{H}g(\alpha)$$

Note that changing variables $\beta = \alpha + \varepsilon\gamma$, we have

$$\int_\mathbb{R} \frac{g(\beta)z'(\beta)}{z(\alpha) - i\varepsilon z'(\alpha) - z(\beta)}\, d\beta = \int_\mathbb{R} \frac{\varepsilon g(\alpha+\varepsilon\gamma)z'(\alpha+\varepsilon\gamma)}{z(\alpha) - i\varepsilon z'(\alpha) - z(\alpha+\varepsilon\gamma)}$$

At the mean time, we also have

$$\int_{|\beta-\alpha|>\varepsilon} \frac{g(\beta)z'(\beta)}{z(\alpha) - z(\beta)}\, d\beta = \int_{|\gamma|\geqslant 1} \frac{\varepsilon g(\alpha+\varepsilon\gamma)z'(\alpha+\varepsilon\gamma)}{z(\alpha) - z(\alpha+\varepsilon\gamma)}\, d\gamma$$

Therefore,

$$\begin{aligned}
&2\pi i G(z(\alpha) - i\varepsilon z'(\alpha)) - \pi i \mathfrak{H}g(\alpha) \\
&= \int_\mathbb{R} \frac{g(\beta)z'(\beta)}{z(\alpha) - i\varepsilon z'(\alpha) - z(\beta)}\, d\beta - \int_{|\beta-\alpha|>\varepsilon} \frac{g(\beta)z'(\beta)}{z(\alpha) - z(\beta)}\, d\beta \\
&= \int_\mathbb{R} \frac{\varepsilon g(\alpha+\varepsilon\gamma)z'(\alpha+\varepsilon\gamma)}{z(\alpha) - i\varepsilon z'(\alpha) - z(\alpha+\varepsilon\gamma)} - \int_{|\gamma|\geqslant 1} \frac{\varepsilon g(\alpha+\varepsilon\gamma)z'(\alpha+\varepsilon\gamma)}{z(\alpha) - z(\alpha+\varepsilon\gamma)}\, d\gamma \\
&= \int_\mathbb{R} g(\alpha+\varepsilon\gamma)\left(\frac{z'(\alpha+\varepsilon\gamma)}{\gamma\frac{z(\alpha)-z(\alpha+\varepsilon\gamma)}{\varepsilon\gamma} - iz'(\alpha)} - \frac{1}{\gamma}\frac{z'(\alpha+\varepsilon\gamma)}{\frac{z(\alpha)-z(\alpha+\varepsilon\gamma)}{\varepsilon\gamma}}1_{\{|\gamma|\geqslant 1\}}\right)\, d\gamma
\end{aligned}$$

Sending $\varepsilon \to 0$ and applying dominated convergence theorem, which is applicable since z parametrizes a Lipschitz curve, the above equation goes to

$$-g(\alpha)\int_\mathbb{R} \frac{1}{\gamma+i} - \frac{1}{\gamma}1_{\{|\gamma|\geqslant 1\}}\, d\gamma = \pi i g(\alpha)$$

which means

$$2\pi i \lim_{\varepsilon\to 0} G(z(\alpha) - i\varepsilon z'(\alpha)) - \pi i \mathfrak{H} g(\alpha) = \pi i g(\alpha)$$

so we have the desired conclusion $g \in C_c^\infty(\mathbb{R})$. □

Having shown this lemma, we want to extend our result from $C_c^\infty(\mathbb{R})$ to $L^p(\mathbb{R})$, where $1 < p < \infty$. We define

$$K_\Sigma(x, y) := \frac{1}{\pi i} \frac{1}{z(x) - z(y)}$$

and have the following theorem by Calderon.

Theorem 11.4 (Calderon, David)**.** *A singular integral operator associated with K_Σ is a Calderon-Zygmund operator.*

Since z' is bounded above, the above theorem immediately gives us the following important corollary, which is worth being called a theorem.

Theorem 11.5. *$\mathfrak{H}$ is a bounded operator $L^p \to L^p$, where $1 < p < \infty$.*

At this point, we notice that we are done if we can show that for any $g \in L^p(\mathbb{R})$, the (non-tangential) limit

$$\lim_{z\to z(\alpha)} G(z) := \lim_{z\to z(\alpha)} \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{g(\beta)z'(\beta)}{z - z(\beta)}\, d\beta \tag{11.6}$$

exists for every $\alpha \in \mathbb{R}$, since we know by Theorem 11.5 that the right hand side of (11.3) is well-behaved if the input is in $L^p(\mathbb{R})$. We wish to apply the standard density argument in harmonic analysis, so we need to define a non-tangential maximal function related to the convergence in (11.6).

For simplicity of presentation, we shall do the case where the Lipschitz curve Σ is just the real line $\mathbb{R}$, since the proof for general case is similar. Henceforth $z(x) = x$ for every $x \in \mathbb{R}$, and we denote

$$\mathbb{H} f(\alpha) = \frac{1}{\pi i}\ p.v. \int \frac{f(\beta)}{\alpha - \beta}\, d\beta$$

We first define several things that shall be used soon.

Definition 11.7. For $t > 0$, we define the Poisson kernel on $\mathbb{R}^2$ as

$$P_t(x) = \frac{t}{x^2 + t^2}$$

and the conjugate Poisson kernel on $\mathbb{R}^2$ as

$$Q_t(x) = \frac{x}{x^2 + t^2}$$

Remark 11.8. We have this terminology since $P_t(x) + iQ_t(x)$ is a holomorphic function on the lower half plane, with respect to the variable $z = x - it$.

Definition 11.9 (Non-tangential Maximal Functions)**.** For $\alpha \in \mathbb{R}$ and $0 < \gamma < \frac{\pi}{2}$, we first define the cone

$$\Gamma_\gamma(\alpha) := \{(x, -t) : x \in \mathbb{R}, t > 0,\ |x - \alpha| \cdot \tan(\gamma) < t\}$$

Then we define

$$Mf(\alpha) := \sup_{\Gamma_\gamma(\alpha)} \left| \int_{\mathbb{R}} \frac{g(\beta)}{z - \beta}\, d\beta \right|$$

We also define the auxiliary maximal functions

$$N_1 f(\alpha) := \sup_{(x,-t)\in\Gamma_\gamma(\alpha)} |P_t * f(x)|$$

$$N_2 f(\alpha) := \sup_{(x,-t)\in\Gamma_\gamma(\alpha)} |Q_t * f(x)|$$

Our goal is to show that M is a bounded operator $L^p \to L^p$ for $1 < p < \infty$, and we will establish this by first showing that N_1 and N_2 are both bounded operator on $L^p \to L^p$.

Proposition 11.10. *N_1 defines a bounded operator $L^p \to L^p$, for any $1 < p < \infty$.*

A discussion of Proposition 11.10 can be found in [2]. The key ingredient to the proof is the following classical lemma in harmonic analysis. The proof of the lemma can also be found in [2].

Lemma 11.11. *Suppose ϕ is dominated by a positive integrable radially decreasing function ψ. Then there is some constant $C > 0$ only depending on ψ such that for any $f \in L^p$ we have*

$$\sup_{t>0} |\phi_t * f(x)| \leqslant C f^*(x)$$

where f^* is the Hardy-Littlewood maximal function of f and

$$\phi_t(x) := \frac{1}{t}\phi\left(\frac{x}{t}\right)$$

We now establish the boundedness of N_2.

Proposition 11.12. *N_2 defines a bounded operator $L^p \to L^p$, for any $1 < p < \infty$.*

Proof. We first assume $f \in \mathcal{S}(\mathbb{R})$. In [2] the author directly computes that

$$Q_t * f = P_t * (\mathbb{H} f) \tag{11.13}$$

for every $t > 0$ using Fourier transform. Note that this implies that (11.13) holds for every $f \in L^p$, since $\mathbb{H}$ is bounded on L^p, convolution by P_t is bounded on L^p, and also $Q_t * f_n \to Q_t * f$ pointwisely for any sequence $f_n \to f$ in L^p by Holder's inequality. Then for any $t > 0$, $\alpha \in \mathbb{R}$, $0 < \gamma < \pi/2$ and $(x, -t) \in \Gamma_\gamma(\alpha)$,

$$Q_t * f(x) = P_t * (\mathbb{H} f)(x)$$

and thus

$$N_2 f(x) \leqslant N_1(\mathbb{H} f)(x)$$

Then L^p boundedness of both N_1 and $\mathbb{H}$ gives us the desired result. $\square$

Now we consider the operator M. Note that for $x - it$ in $\Gamma_\gamma(\alpha)$,

$$\begin{aligned}
\left| \int_{\mathbb{R}} \frac{g(\beta)}{x-\beta-it}\, d\beta \right| &= \left| \int_{\mathbb{R}} \frac{g(\beta)(\beta - it)}{(x-\beta)^2+t^2}\, d\beta \right| \\
&\leqslant \left| \int_{\mathbb{R}} \frac{g(\beta)\beta}{(x-\beta)^2+t^2}\, d\beta \right| + \left| \int_{\mathbb{R}} \frac{g(\beta)t}{(x-\beta)^2+t^2}\, d\beta \right| \\
&= |Q_t * f(x)| + |P_t * f(x)| \\
&\leqslant N_1 f(\alpha) + N_2 f(\alpha)
\end{aligned}$$

So we have

$$Mf(\alpha) \leqslant N_1 f(\alpha) + N_2 f(\alpha)$$

which leads to our desired result.

Proposition 11.14 (Boundedness of Non-tangential maximal function)**.** *M defines a bounded operator $L^p \to L^p$, for any $1 < p < \infty$.*

Now our proof is complete.

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