NOTES ON PARTIAL DIFFERENTIAL EQUATIONS

ZHENGJUN LIANG

Abstract. These notes record some foundational knowledge for PDE research and are evolving over time. They are supposed to be high-level sketches, so intuitions and main ideas are emphasized, and sometimes only references are given for detailed proofs. At the meantime, examples are computed, and proofs not easily found in literature are supplied. (Last Updated: Jan 8, 2024.)

Contents

1. Preliminaries

The following integration by parts formula is used frequently. Here U is open.

$$\int_U u \operatorname{div}(V) = \int_{\partial U} uV \cdot n - \int_U \nabla u \cdot V \tag{1.1}$$

Another thing that we use often is the weighted Cauchy-Schwarz inequality

$$2ab \leqslant \frac{a^2}{\varepsilon} + \varepsilon b^2 \tag{1.2}$$

here $\varepsilon > 0$.

2. Gronwall Inequality and Bootstrap Argument

Theorem 2.1 (Gronwall's Inequality, Differential Form)**.** *Let $u : [t_0, t_1] \to \mathbb{R}^+$ be absolutely continuous and non-negative, and suppose u obeys the differential inequality*

$$\partial_t u(t) \leqslant B(t)u(t) + C(t) \tag{2.2}$$

for a.e. $t \in [t_0, t_1]$, where $B : [t_0, t_1] \to \mathbb{R}^+$ is continuous and non-negative. Then we have

$$u(t) \leqslant \exp\left(\int_{t_0}^t B(s)\ ds\right)\left(u(t_0) + \int_{t_0}^t C(s)\ ds\right) \tag{2.3}$$

for all $t \in [t_0, t_1]$.

Theorem 2.4 (Gronwall's Inequality, Integral Form)**.** *Let $u : [t_0, t_1] \to \mathbb{R}^+$ be absolutely continuous and non-negative, and suppose u obeys the integral inequality*

$$u(t) \leqslant A + \int_{t_0}^t B(s)u(s)\ ds + \int_{t_0}^t C(s)\ ds \tag{2.5}$$

for all $t \in [t_0, t_1]$, where $B : [t_0, t_1] \to \mathbb{R}^+$ is continuous and non-negative. Then we have

$$u(t) \leqslant \exp\left(\int_{t_0}^t B(s)\ ds\right)\left(u(t_0) + \int_{t_0}^t C(s)\ ds\right) \tag{2.6}$$

for all $t \in [t_0, t_1]$.

Theorem 2.7 (Abstract Bootstrap Principle)**.** *Let $[0, T]$ be a closed (time) interval. $H(t), C(t)$ are two statements that we heuristically understand as "hypothesis" and "conclusion". Suppose*

1. *(Hypothesis implies conclusion) If $H(t)$ is true, then $C(t)$ is true.*
2. *(Conclusion stronger than hypothesis) If $C(t_0)$ is true, then there is an interval U containing t_0 such that $H(t)$ is true for $t \in U$.*
3. *(Conclusion is closed) If t_n is a sequence in $[0, T]$ converging to t, $C(t_n)$ is true for every n, then $C(t)$ is true.*
4. *(Hypothesis is non-vacuous) $H(t)$ is true for some $t \in [0, T]$.*

Then $C(t)$ is true for all $t \in [0, T]$.

The proof is a standard connectivity argument (as can be found in [6]), namely showing that the set of t on which $C(t)$ is true is non-empty, open, and closed in $[0, T]$, so it must be $[0, T]$ itself.

3. Laplace Equation

We study the Laplace equation

$$-\Delta u = 0 \tag{3.1}$$

and Poisson's equation

$$-\Delta u = f \tag{3.2}$$

where u and f are nice enough functions. When we solve the equations on a domain U, it is common to assume that $u \in C^2(U) \cap C(\overline{U})$ and $f \in C(\overline{U})$.

Definition 3.3 (Harmonic Function, First Version)**.** A function $u \in C^2(U) \cap C(\overline{U})$ that solves the Laplace equation (3.1) is called a harmonic function.

3.1. Fundamental Solution.

We begin with an attempt to find a radially symmetric solution u of the Laplace equation (3.1) in $U = \mathbb{R}^n$, i.e.

$$u(x) = v(r)$$

where $r = |x|$. Direct computation yields

$$v(r) = \begin{cases} b \log r + c & n = 2 \\ \frac{b}{r^{n-2}} + c & n \geqslant 3 \end{cases}$$

This motivates us to define

Definition 3.4 (Fundamental Solution of Laplace Equation)**.** The function

$$\Phi(x) = \begin{cases} -\frac{1}{2\pi} \log |x| & n = 2 \\ \frac{1}{n(n-2)\omega(n)} \frac{1}{|x|^{n-2}} & n \geqslant 3 \end{cases}$$

defined for $x \in \mathbb{R}^n$, $x \neq 0$, is called the fundamental solution of Laplace equation. Here $\omega(n)$ refers to the volume of the unit ball in $\mathbb{R}^n$.

The fundamental solution can also be found through Fourier transform. At least formally, we can take Fourier transform of the equation

$$\Delta u = \delta$$

to obtain

$$4\pi|\xi|^2 \widehat{u}(\xi) = 1$$

and thus

$$\widehat{u}(\xi) = \frac{1}{4\pi|\xi|^2}$$

Now taking inverse Fourier transform we get the desired result, at least formally. Note that we have the estimates

$$|\nabla u(x)| \leqslant \frac{C}{|x|^{n-1}}, \quad |D^2 u(x)| \leqslant \frac{C}{|x|^n}$$

The main reason we study the fundamental solution is that it will help us solve the Poisson's equation, as seen in the following result.

Theorem 3.5 (Solving Poisson's Equation)**.** *Let $f \in C^2_c(\mathbb{R}^n)$, and we define*

$$u(x) := \Phi * f(x) := \int_{\mathbb{R}^n} \Phi(x-y) f(y) \, dy$$

Then $u \in C^2(\mathbb{R}^n)$ and $-\Delta u = f$ in $\mathbb{R}^n$.

We discuss heuristically what happens. We can compute that "$\Delta\Phi(x) = \delta$", and thus

$$\text{“}\Delta(\Phi * f)(x) = (\Delta\Phi * f)(x) = (\delta * f)(x) = f(x)\text{”}$$

3.2. **Maximum Principle.** Harmonic functions enjoys a **mean value property**, which says that the average of a harmonic function u over a sphere $\partial B(x,r)$ or over the ball $B(x,r)$ is equal to $u(x)$, provided that $B(x,r) \subset U$. The statement is

Theorem 3.6 (Mean-Value Formula)**.** *A function $u \in C^2(U)$ is harmonic if and only if*

$$u(x) = \frac{1}{|\partial B(x,r)|}\int_{\partial B(x,r)} u \, dS = \frac{1}{|B(x,r)|}\int_{B(x,r)} u \, dy$$

for each ball $B(x,r) \subset U$.

This mean value property gives rise to remarkable consequences, such as the maximum principle.

Theorem 3.7 (Strong Maximum Principle)**.** *Suppose $u \in C^2(U) \cap C(\overline{U})$ is harmonic within U. Then*

$$\max_{\overline{U}} u = \max_{\partial U} u$$

Furthermore, if U is connected and there is a point $x_0 \in U$ such that

$$u(x_0) = \max_{\overline{U}} u$$

then u is constant within $\overline{U}$.

The proof of the strong maximum principle is a standard connectivity argument using the mean value formula.

Theorem 3.8 (Regularity)**.** *If $u \in C(U)$ satisfies the mean value property for each $B(x,r) \subset U$, then in fact $u \in C^\infty(U)$. In particular, harmonic functions are smooth.*

To prove the theorem, one consider the standard mollification $u^\varepsilon := u * \eta_\varepsilon$ in $U_\varepsilon := \{x \in U : \operatorname{dist}(x, \partial U) > \varepsilon\}$, and show that in fact $u \equiv u^\varepsilon$ on U_ε.

Theorem 3.9 (Harnack Inequality)**.** *Let u be a non-negative harmonic function on U. For each connected open set $V \subset\subset U$, there is a positive constant C, depending only on V, such that*

$$\sup_V u \leqslant C \inf_V u$$

for all non-negative harmonic functions u in U.

The inequality asserts that the values of a non-negative harmonic function are all comparable on U. The proof utilizes mean value property and compactness of $\overline{V}$.

3.3. **Uniqueness.** Suppose $f \in C(\overline{U})$ with $f = g$ on ∂U. We can in fact show that solution to the Poisson equation

$$\Delta u = f \text{ on } U, \quad u = g \text{ on } \partial U$$

is unique. We present two different proofs, one by maximum principle and one by energy method.

Let u, u' be two such solutions. Then $w := u - u'$ satisfies

$$\Delta w = 0 \text{ on } U, \quad w = 0 \text{ on } \partial U$$

and it follows from the maximum principle that $w \equiv 0$ on $\overline{U}$.

As for the energy method, we consider the energy

$$E(w) := \int_U |\nabla w|^2 \, dx$$

and integration by parts gives

$$E(w) := -\int_U \Delta w \cdot w \, dx = 0$$

where the boundary term vanishes because $w = 0$ on ∂U. Since ∇w is continuous, we must actually have $\nabla w = 0$. Then we get $w \equiv 0$ noting that $w = 0$ on ∂U.

3.4. **Green's Representation Formula.** Assume now that $U \subset \mathbb{R}^n$ is open, bounded, and ∂U is C^1. We propose a general representation formula for the solution of Poisson's equation

$$-\Delta u = f \quad \text{in } U$$

subject to the boundary condition

$$u = g \quad \text{on } \partial U$$

Definition 3.10. For $x, y \in U$ and $x \neq y$, Green's function for the region U is

$$G(x,y) := \Phi(y-x) - \phi^x(y)$$

where Φ is the fundamental solution as defined above, and ϕ^x solves the boundary value problem

$$\begin{cases} \Delta \phi^x(y) = 0 & \text{in } U \\ \phi^x(y) = \Phi(y-x) & \text{on } \partial U \end{cases}$$

With the Green's function defined, we have the following representation formula:

Theorem 3.11. *If $u \in C^2(\overline{U})$ solves the Poisson equation with boundary data as above, then*

$$u(x) = -\int_{\partial U} g(y) \frac{\partial G}{\partial \nu}(x,y) \, dS(y) + \int_U f(y) G(x,y) \, dy$$

for $x \in U$.

4. Heat Equation

This section is devoted to study heat equation of the form

$$\partial_t u - \Delta u = f \tag{4.1}$$

which is the prototype of **parabolic equations**.

4.1. **Fundamental Solution.** As in the study of Laplace equation, we begin by looking for a radially symmetric solution that behaves well under scaling. This motivates us to define the following fundamental solution:

Definition 4.2 (Fundamental Solution of the Heat Equation)**.** The function

$$\Phi(x,t) = \frac{1}{(4\pi t)^{n/2}} e^{-|x|^2/4t}$$

is called the fundamental solution of the heat equation.

The fundamental solution can also be found using Fourier transform. Formally taking Fourier transform of the heat equation, we get

$$\partial_t \widehat{u}(\xi,t) + 4\pi^2|\xi|^2\widehat{u}(\xi,t) = 0$$

This is an ODE, of which

$$\widehat{u}(\xi,t) = \exp\big(-4\pi^2|\xi|^2 t\big)$$

is a solution. Taking inverse Fourier transform we obtain the fundamental solution up to multiplication by a constant.

The constants as in the definition of the fundamental solution are chosen to satisfy the following normalization:

Lemma 4.3. *For each time $t > 0$,*

$$\int_{\mathbb{R}^n} \Phi(x,t)\ dx = 1$$

The reason we study the fundamental solution for the heat equation is that it will help us solve the heat equation with a given initial data, as suggested by the following theorem:

Theorem 4.4. *Let $g \in L^\infty(\mathbb{R}^n) \cap C(\mathbb{R}^n)$, and we define*

$$u(x,t) := \Phi * g = \int_{\mathbb{R}^n} \Phi(x-y,t)g(y)\ dy$$

Then $u \in C^\infty(\mathbb{R}^n \times (0,\infty))$, and u satisfies

$$\begin{cases} \partial_t u - \Delta u = 0 & \text{for } x \in \mathbb{R}^n,\ t > 0 \\ \lim_{(x,t)\to(x_0,0)} u(x,t) = g(x_0) & \text{for } x_0 \in \mathbb{R}^n \end{cases}$$

4.2. **Maximum Principle.**

Theorem 4.5 (Strong Maximum Principle)**.** *Assume $u \in C^2(U_T) \cap C(\overline{U_T})$ is a solution to the heat equation on U_T. Then*

(1) We have

$$\max_{\overline{U_T}} |u| = \max_{\Gamma_T} |u|$$

(2) Moreover, if

$$\max_{\overline{U_T}} |u| = |u(x_0,t_0)|$$

for some $(x_0,t_0) \in U_T$, then u is in fact constant on $\overline{U_T}$.

A proof can be found in [4]. It utilizes a mean-value property for heat equations.

The strong maximum principle also exemplifies **infinite speed of propagation**, which is a common feature of parabolic equations. More specifically, suppose we start with an initial data $u(0) = g \geqslant 0$, then as soon as $t > 0$, $u(t)$ is **everywhere strictly greater than** 0 by strong maximum principle. Of course, in the case of heat equation we can also see infinite speed of propagation directly through the explicit solution formula given by the fundamental solution, but the strong maximum principle holds in a more general setting.

Hyperbolic equations such as the wave equation have no such maximum principles, and in contrast demonstrate **finite speed of propagation**, as seen in Theorem 5.7.

4.3. **Duhammel's Formula.** We now use Duhammel's principle to solve the inhomogeneous problem

$$\begin{cases} \partial_t u - \Delta u = f & \text{in } \mathbb{R}^n \times (0,\infty) \\ u(\cdot,0) = 0 & \text{on } \mathbb{R}^n \times \{t=0\} \end{cases} \tag{4.6}$$

where we assume $f \in C_c^2(\mathbb{R}^n)$ for convenience. The trick is to define

$$u(x,t;s) := \int_{\mathbb{R}^n} \Phi(x-y,t-s) f(y,s)\ dy$$

where s can be regarded as a parameter. Then $u(x,t;s)$ solves the initial value problem

$$\begin{cases} \partial_t u(\cdot;s) - \Delta u(\cdot;s) = 0 & \text{in } \mathbb{R}^n \times (s,\infty) \\ u(\cdot;s) = f(\cdot;s) & \text{on } \mathbb{R}^n \end{cases}$$

We let

$$u(x,t) = \int_0^t u(x,t;s)\ ds = \int_0^t \frac{1}{(4\pi(t-s))^{n/2}} \int_{\mathbb{R}^n} \exp\left(-\frac{|x-y|^2}{4(t-s)}\right) f(y,s)\ dyds$$

and one can verify that u actually solves the inhomogeneous problem. It is clear that $u(\cdot,0) = 0$. Moreover, for $t > 0$,

$$\Delta u = \int_0^t \Delta u(x,t;s)\ ds = 0$$

since Φ is harmonic for $t - s > 0$, and

$$\partial_t u(x,t) = \lim_{s \to t^-} u(x,t;s) = f(x,t)$$

as desired.

4.4. **Uniqueness.** Suppose $f = C(\overline{U_T})$ and $g \in C(\overline{U})$. As for Laplace equation, the initial value problem for the heat equation has a unique classical solution. On the one hand, this is an easy consequence of the strong maximum principle. On the other hand, one can use energy method to prove that $u \equiv 0$ if f and g are both identically zero. Consider the energy

$$E(t) := \frac{1}{2}\|u(t)\|_{L^2(U)}^2$$

Then we have

$$\frac{dE}{dt} = \int_U \partial_t u \cdot u = \int_U u\Delta u = -\int_U |\nabla u|^2 \leqslant 0$$

The last equality follows from integration by parts, and boundary value vanishes because we assume $u \equiv 0$ on ∂U. On the other hand, $u(0) = g \equiv 0$, so $E(0) = 0$, and thus $E(t) = 0$ for all $t \in [0,T]$ since $E(t)$ is always non-negative. Hence $\nabla u \equiv 0$ and $u \equiv 0$ on U_T by zero boundary conditions.

5. Wave Equation

We now consider the wave equation

$$u_{tt} - \Delta u = 0 \tag{5.1}$$

with initial data

$$u(\cdot,0) = g, \quad u_t(\cdot,0) = h$$

At times we will use the D'Alembertian $\square$ to denote $\partial_t^2 - \Delta$.

5.1. **Solution in 1D.** We notice that

$$(\partial_t^2 - \partial_x^2)\phi(x \pm t) = 0$$

so we seek a solution of the form

$$u = \phi(x+t) + \psi(x-t)$$

Setting $t = 0$, we must have

$$\phi(x) + \psi(x) = g, \quad \phi'(x) - \psi'(x) = h$$

It follows that

$$\phi'(x) = \frac{1}{2}(g'(x) + h(x)), \quad \psi'(x) = \frac{1}{2}(g'(x) - h(x))$$

and thus

$$\phi(x) = \frac{1}{2}g(x) + \frac{1}{2}\int_0^x h(s)\ ds + C_1$$
$$\psi(x) = \frac{1}{2}g(x) - \frac{1}{2}\int_0^x h(s)\ ds + C_2$$

We have $C_1 + C_2 = 0$ since $\phi + \psi = g$, so

$$u(x) = \frac{1}{2}[g(x+t) + g(x-t)] + \frac{1}{2}\int_{x-t}^{x+t} h(s)\ ds \tag{5.2}$$

gives a solution. This is called the D'Alembert's formula.

5.2. **Solution in Higher Dimensions.** In higher dimensions, we look at the mean value of u, g, h over the sphere $B(x,r)$. Therefore, we define

$$U(x,r,t) := \frac{1}{|\partial B(x,r)|}\int u(y,t)\ dS(y)$$
$$G(x,r) := \frac{1}{|\partial B(x,r)|}\int g(y)\ dS(y)$$
$$H(x,r) := \frac{1}{|\partial B(x,r)|}\int h(y)\ dS(y)$$

Then direct calculation gives an equation that U satisfies:

Lemma 5.3 (Euler-Poisson-Darboux Equation)**.** *Fix $x \in \mathbb{R}^n$, we have*

$$U_{tt} - U_{rr} - \frac{n-1}{r}U_r = 0 \quad \text{in } \mathbb{R}_+ \times (0,\infty) \tag{5.4}$$

with initial data

$$U = G,\ U_t = H \quad \text{on } \mathbb{R}_+ \times \{t = 0\}$$

We then establish an explicit solution formula for the $n = 3$ case, called Kirchhoff's formula. Denoting $\tilde{U} := rU$, $\tilde{G} := rG$, and $\tilde{H} := rH$, we have

$$\tilde{U}_{tt} - \tilde{U}_{rr} = 0 \quad \text{in } \mathbb{R}_+ \times (0,\infty)$$

with intial data

$$\tilde{U} = \tilde{G}, \quad \tilde{U}_t = \tilde{H} \quad \text{on } \{r = 0\} \times (0,\infty)$$

and boundary data

$$\tilde{U} = 0 \quad \text{on } \{r = 0\} \times (0,\infty)$$

The D'Alembert's formula (5.2) gives us

$$\tilde{U}(x;r,t) = \frac{1}{2}[\tilde{G}(r+t) - \tilde{G}(t-r)] + \frac{1}{2}\int_{-r+t}^{r+t} \tilde{H}(y)\ dy$$

Dividing the above equation by r and sending $r \to 0$ we get

$$u(x,t) = \tilde{G}'(t) + \tilde{H}(t)$$

Straightforward computation then yields

$$u(x,t) = \frac{1}{|\partial B(x,t)|}\int_{\partial B(x,t)} th(y) + g(y) + \nabla g(y)\cdot(y-x)\ dS(y) \tag{5.5}$$

which is the Kirchhoff's formula.

To establish the $n = 2$ case, the strategy is regarding a function taking input in $\mathbb{R}^2$ as a function taking input in $\mathbb{R}^3$. We omit the details here.

5.3. **Energy Methods.** We use energy methods to prove two results: uniqueness of the Cauchy problem and finite speed of propagation.

Theorem 5.6 (Uniqueness)**.** *Let U be an open domain in $\mathbb{R}^n$. Then the boundary value problem*

$$u_{tt} - \Delta u = 0$$

on U with initial data

$$u(\cdot,0) = g, \quad u_t(\cdot,0) = h$$

and boundary value

$$u = f \quad \text{on } \partial U$$

admits a unique classical solution.

Proof. By linearity, it suffices to show that a solution w to the wave equation with zero boundary and initial data vanishes everywhere. Consider

$$E(t) := \frac{1}{2}\int_U w_t^2 + |\nabla w|^2\ dx$$

Then

$$\begin{aligned}\frac{d}{dt}E(t) &= \int_U w_t w_{tt} + (\nabla w_t)\cdot(\nabla w)\ dx \\ &= \int_U w_t(w_{tt} - \Delta w)\ dx \\ &= 0\end{aligned}$$

where the second equality follows from integration by parts. By initial condition, $E(0) = 0$, and thus $E(t) = 0$ for all t. In particular, $\nabla w \equiv 0$ on U, and we get $w \equiv 0$ on U by zero boundary condition. $\square$

Theorem 5.7 (Finite Speed of Propagation)**.** *Let u be a solution to the homogeneous wave equation. Suppose $u \equiv 0$ on $B(x_0,t_0)$. Then $u \equiv 0$ in the cone $\{(x,t) : |x - x_0| \leqslant t_0 - t\}$.*

Proof. We define

$$E(t) := \frac{1}{2}\int_{B(x_0,t_0-t)} u_t^2 + |\nabla u|^2 \, dx$$

and differentiating the energy in t gives

$$\begin{aligned}\frac{d}{dt}E(t) &= \int_{B(x_0,t_0-t)} u_t u_{tt} + (\nabla u_t)\cdot(\nabla u)\, dx - \frac{1}{2}\int_{\partial B(x_0,t_0-t)} u_t^2 + |\nabla u|^2 \, dS \\ &= \int_{\partial B(x_0,t_0-t)} u_t \frac{\partial u}{\partial \nu} - \frac{1}{2}u_t^2 - \frac{1}{2}|\nabla u|^2 \, dS \\ &\leqslant 0\end{aligned}$$

where the second equality follows from integration by parts and the last equality follows from weighted Cauchy-Schwarz inequality. Since $u \equiv 0$ on $B(x_0, t_0)$, we have $E(0) = 0$, and thus $E(t) = 0$ for all t. In particular, $u(t)$ is a constant on $B(x_0, x_0 - t)$, and we know this constant must be 0 since $u \equiv 0$ on $B(x_0, t_0)$. □

6. Sobolev Spaces and Inequalities

The theorems in this section are standard Sobolev embeddings, and proof can be found on [4]. For a Fourier analytic approach, interested readers can also look at [1].

Theorem 6.1 (Gagliardo-Nirenberg)**.** *Let $1 \leqslant p < n$. For $u \in C_c^1(\mathbb{R}^n)$, we have*

$$\|u\|_{L^{p^*}} \leqslant C\|\nabla u\|_{L^p}$$

where $\frac{n}{p} - \frac{n}{p^*} = 1$.

Remark 6.2. The value of p^* can be seen through scaling. We define

$$u_\lambda(x) = u(\lambda x)$$

and test the inequality for u_λ.

As a corollary, we present the following Poincaré inequality that is of independent interest.

Theorem 6.3. *Assume U is a bounded open subset of $\mathbb{R}^n$. Suppose $u \in W_0^{1,p}(U)$ for some $1 \leqslant p \leqslant n$. Then*

$$\|u\|_{L^q(U)} \leqslant C\|\nabla u\|_{L^p(U)}$$

for each $q \in [1, p^*]$, and the constant C depends only on p, q, n and U. In particular, for all $1 \leqslant p \leqslant \infty$,

$$\|u\|_{L^p(U)} \leqslant C\|\nabla u\|_{L^p(U)}$$

Theorem 6.4 (Morrey's Inequality)**.** *Let $p > n$, and $\gamma = 1 - \frac{n}{p}$. Then we have*

$$\|u\|_{C^{0,\gamma}(\mathbb{R}^n)} \leqslant C\|u\|_{W^{1,p}(\mathbb{R}^n)}$$

The proof relies crucially on the inequality

$$\frac{1}{|B(x,r)|}\int_{B(x,r)} |u(y) - u(x)| \, dy \leqslant C\int_{B(x,r)} \frac{|\nabla u(y)|}{|y-x|^{n-1}} \, dy$$

for each ball $B(x,r) \subset \mathbb{R}^n$.

Theorem 6.5 (Rellich-Kondrakov Compactness Theorem)**.** *Assume U is a bounded open subset of $\mathbb{R}^d$ and ∂U is C^1. Suppose $1 \leqslant p < n$, then $W^{1,p}$ compactly embedds into $L^q(U)$ for $1 \leqslant q < p^*$.*

Given a bounded sequence $\{u_m\}_m \subset W^{1,p}(U)$, we want to show that it admits a convergent subsequence in $L^q(U)$. By Sobolev extension theorem, we may assume u_m are defined on $\mathbb{R}^n$ and compactly supported in some bounded open V, so we actually show that $\{u_m\}_m$ has a convergent subsequence in $L^q(V)$ given that $\{u_m\}_m \subset W^{1,p}(V)$ is uniformly bounded. Let u_m^ε be the standard mollification, we show that $u_m^\varepsilon \to u_m$ in $L^q(V)$ uniformly in m, and that for each $\varepsilon > 0$, there is a subsequence $\{u_{m_j}^\varepsilon\}_j$ that converges uniformly. Combining everything above, we can show that for every $\delta > 0$, we can find a subsequence $\{u_{m_j}\}_j$ such that

$$\limsup_{j,k\to\infty} \|u_{m_j} - u_{m_k}\|_{L^q(V)} \leqslant \delta$$

We can then let $\delta = 1, \frac{1}{2}, \frac{1}{3}, \ldots$ and extract a convergent subsequence $\{u_{m_l}\}_l \subset L^q(V)$ using a standard diagonal argument.

7. Linear Elliptic Equations

In this section, we study equations of the form

$$Lu = f$$

on a domain U, where

$$Lu(x) = -\sum_{i,j=1}^n a_{ij}(x)u_{x_ix_j}(x) + \sum_{i=1}^n b_i(x)u_{x_i}(x) + c(x)u(x) \tag{7.1}$$

We can alternatively write L in the divergence form

$$Lu = -\mathrm{div}(A\cdot\nabla u) + B\cdot\nabla u + cu$$

so that we can define the bilinear form

$$B[u,v] := \int_U \nabla v(x)\cdot(A(x)\cdot\nabla u(x))\ dx + \int_U v(x)(B(x)\cdot\nabla u(x))\ dx + \int_U c(x)u(x)v(x)\ dx$$

We say that $u \in H_0^1(U)$ is a weak solution if for every $v \in H_0^1(U)$,

$$B[u,v] = \int_U f(x)v(x)\ dx$$

Theorem 7.2 (A Priori Estimates)**.** *Let L be an operator of divergence form, with coefficients $A, B, C \in L^\infty(U)$. Then*

(1) $|B[u,v]| \leqslant \alpha\|u\|_{H_0^1}\|v\|_{H_0^1}$ for some $\alpha > 0$.

(2) $|B[u,v]| \geqslant \beta\|u\|_{H_0^1}^2 - \gamma\|u\|_{L^2}^2$ for some $\beta, \gamma > 0$.

Proof. The first estimate is a direct consequence of Cauchy-Schwarz. The second one exploits ellipticity and Poincaré's inequality. □

Theorem 7.3 (Lax-Milgram Theorem)**.** *Let H be a Hilbert space, and $B : H\times H \to \mathbb{R}$ is a bilinear form such that there are constants $\alpha, \beta > 0$ such that*

$$B[u,v] \leqslant \alpha\|u\|\|v\|$$

and

$$B[u,u] \geqslant \beta\|u\|^2$$

Then, for every linear functional $f : H \to \mathbb{R}$, there is some $u \in H$ such that

$$f(v) = B[u,v]$$

for every $v \in H$.

Remark 7.4. If B is symmetric, then B is in fact an inner product and the conclusion directly follows from Riesz representation theorem. Lax-Milgram is significant primarily because it doesn't have any symmetry assumption on B.

Theorem 7.5 (First Existence Theorem). *There is a number* $\gamma \geqslant 0$ *such that for each* $\mu \geqslant \gamma$ *and each function* $f \in L^2(U)$, *there exists a unique weak solution* $u \in H_0^1(U)$ *to the boundary value problem*

$$\begin{cases} Lu + \mu u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

For the second existence theorem, we need the following properties of compact operators. One of the characterizations of compact operators $H \to H$ is that they are the limits of finite-rank operators $H \to H$ in $L(H, H)$, the space of bounded linear operators on H, under the operator topology. Therefore, we expect compact operators to preserve some nice properties of finite-rank operators.

Theorem 7.6 (Fredholm Alternative). *Let* H *be a Hilbert space, and* $K : H \to H$ *is a compact linear operator. Then*

(1) $N(I - K)$ *is finite dimensional.*
(2) $R(I - K)$ *is closed.*
(3) $R(I - K) = N(I - K^*)^{\perp}$.
(4) $N(I - k) = \{0\}$ *if and only if* $R(I - K) = H$.
(5) $\dim N(I - K) = \dim N(I - K^*)$.

Theorem 7.7 (Second Existence Theorem). *We have the following dichotomy:*

(1) *either for each* $f \in L^2(U)$, *there exists a unique weak solution* u *such that the boundary value problem*
$$\begin{cases} Lu = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$
has a unique solution,
(2) *or there exists a non-trivial solution* $u \not\equiv 0$ *to the boundary value problem*
$$\begin{cases} Lu = 0 & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

Moreover, if the second assertion holds, the space of weak solutions $N \subset H_0^1(U)$ *is finite dimensional, and equals the dimension of solutions* $N^* \subset H_0^1(U)$ *to the dual problem*

$$\begin{cases} L^* u = 0 & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

Eventually, the boundary value problem in (1) has a solution if and only if

$$\langle f, v \rangle_{L^2} = 0$$

for all $v \in N^*$.

Definition 7.8. Let X be a Banach space, and $A : X \to Y$ be a bounded linear operator.

(1) The resolvent set of A is
$$\rho(A) := \{\eta \in \mathbb{R} : (A - \eta I) \text{ is one to one and onto}\}$$

(2) The spectrum of A is

$$\sigma(A) = \mathbb{R} - \rho(A)$$

(3) We say $\lambda \in \sigma(A)$ is an eigenvalue of A if

$$N(A - \lambda I) \neq 0$$

and we write $\sigma_p(A)$ for the collection of eigenvalues of A, called the point spectrum of A.

Theorem 7.9 (Spectral Properties of Compact Operators)**.** *Assume* $\dim H = \infty$ *and* $K : H \to H$ *is compact. Then*

(1) $0 \in \sigma(K)$,
(2) $\sigma(K) - \{0\} = \sigma_p(K) - \{0\}$.
(3) $\sigma_p(K)$ *is either a finite set or a sequence of numbers going to* 0.

Theorem 7.10 (Third Existence Theorem)**.** *There is an at most countable set* Σ*, such that the initial value problem*

$$\begin{cases} Lu + \lambda u = f & \text{in } U \\ u = 0 & \text{on } \partial U \end{cases}$$

has a unique solution for all λ *not in* Σ*. Moreover, if* Σ *is infinite, then* $\Sigma = \{\lambda_k\}_1^\infty$*, where* $\lambda_k \to \infty$ *as* $k \to \infty$.

7.1. **Maximum Principles.** In this section we collect weak and strong maximum principles for elliptic equations. They can also be found on Chapter 6 of [4].

Theorem 7.11 (Weak Maximum Principle, $c = 0$)**.** *Suppose* $u \in C^2(U) \cap C(\overline{U})$ *and* $c = 0$*. Then*

(1) If $Lu \leqslant 0$ *in* U*, then* $\max_{\overline{U}} u = \max_{\partial U} u$.
(2) If $Lu \geqslant 0$ *in* U*, then* $\min_{\overline{U}} u = \min_{\partial U} u$.

Theorem 7.12 (Weak Maximum Principle, $c \geqslant 0$)**.** *Suppose* $u \in C^2(U) \cap C(\overline{U})$ *and* $c \geqslant 0$*. Then*

(1) If $Lu \leqslant 0$ *in* U*, then* $\max_{\overline{U}} u \leqslant \max_{\partial U} u^+$.
(2) If $Lu \geqslant 0$ *in* U*, then* $\min_{\overline{U}} u \leqslant -\max_{\partial U} u^-$.

Theorem 7.13 (Hopf Lemma)**.** *Assume* $u \in C^2(U) \cap C(\bar{(U)})$ *and* $c \equiv 0$ *in* U*. Suppose further that*

$$Lu \leqslant 0 \quad \text{in } U$$

and that there exists a point $x_0 \in \partial U$ *such that*

$$u(x_0) > u(x) \quad \text{for all } x \in U$$

If U *satisfies the interior ball condition at* x_0*, that is, there exists a ball* $B \subset U$ *such that* $x_0 \in \partial B$*, then we must have*

$$\frac{\partial u}{\partial \nu}(x_0) > 0$$

where ν *is the outward normal of* ∂U *at* x_0*. If* $c \geqslant 0$*, the above conclusion holds provided* $u(x_0) \geqslant 0$.

Theorem 7.14 (Strong Maximum Principle)**.** *Suppose* $c = 0$*,* U *is bounded, open and connected. Assume also that* $u \in C^2(U) \cap C(\overline{U})$*. If*

(1) $Lu \leqslant 0$, and there is an interior point that achieves the maximum on $\overline{U}$, or
(2) $Lu \geqslant 0$, and there is an interior point that achieves the minimum on $\overline{U}$,

Then u is constant on $\overline{U}$.

7.2. **Heuristic Discussion of Regularity.** If we look at the Poisson equation $\Delta u = f$ on U with $u = 0$ on ∂U, then

$$\int_U |f|^2 = \int_U |\Delta u|^2$$

If we formally integrate by parts, the above equation actually equals to

$$\int_U |D^2 u|^2$$

So the L^2 norm of the second derivative of u can be estimated in terms of the L^2 norm of f. Therefore, if f has enough regularity, say $H^\infty(U)$, then we can hope that a weak solution $u \in H_0^1(U)$ to the above equation is actually in $H^\infty(U)$. To carry out the above calculation rigorously, we would need to use difference quotients.

8. Linear Parabolic Equations

We construct weak solutions by Galerkin method. Let $\{w_k\}_k$ be an orthonormal basis in $L^2(U)$ and an orthogonal basis in $H_0^1(U)$. We look for $u_m : [0,T] \to H_0^1(U)$ such that

$$u_m(t) = \sum_{k=1}^m d_m^k(t) w_k$$

where d_m^k is constructed such that

$$d_m^k(0) = \langle g, w_k \rangle \tag{8.1}$$

and

$$\langle u_m', w_k \rangle + B[u_m, w_k; t] = \langle f, w_k \rangle \tag{8.2}$$

Note that (8.2) is equivalent to

$$(d_m^k)'(t) + \sum_1^m B[w_l, w_k; t] d_m^l(t) = \langle f, w_k \rangle \tag{8.3}$$

for $k = 1, ..., m$. Note that this is a system of ODEs. Together with (8.1), existence and uniqueness of d_m^k's are guaranteed by the fundamental theorem of ODEs.

Theorem 8.4 (Energy Estimates)**.** *We have*

$$\max_{0 \leqslant t \leqslant T} \|u_m(t)\|_{L^2(U)} + \|u_m\|_{L^2([0,T];H_0^1(U))} + \|u_m'\|_{L^2([0,T];H^{-1}(U))} \leqslant C(\|g\|_{L^2(U)} + \|f\|_{L^2([0,T];L^2(U))})$$

The intuition of this energy estimate can be seen through the inhomogeneous heat equation

$$\partial_t u - \Delta u = f$$

with $u(0) = g$. We multiply both sides with u and integrate in time to have

$$\frac{d}{dt}\|u\|_{L^2(U)}^2 + \|\nabla u\|_{L^2(U)}^2 = \int_U fu$$

Using weighted Cauchy-Schwarz inequality (1.2), we have

$$\frac{d}{dt}\|u\|^2_{L^2(U)} + \|\nabla u\|^2_{L^2(U)} \leqslant C\|f\|^2_{L^2(U)}$$

so Gronwall's inequality gives

$$\max_{0\leqslant t\leqslant T}\|u(t)\|_{L^2(U)} + \|u\|_{L^2([0,T];H^1_0(U))} \leqslant C(\|g\|_{L^2(U)} + \|f\|_{L^2([0,T];L^2(U))})$$

Meanwhile, note that

$$\begin{aligned}\|\partial_t u\|_{L^2([0,T];H^{-1}(U))} &\leqslant \|\Delta u\|_{L^2([0,T];H^{-1}(U))} + \|f\|_{L^2([0,T];H^{-1}(U))}\\ &\leqslant \|u\|_{L^2([0,T];H^1_0(U))} + \|f\|_{L^2([0,T];L^2(U))})\\ &\leqslant C(\|g\|_{L^2(U)} + \|f\|_{L^2([0,T];L^2(U))})\end{aligned}$$

so we get the desired energy estimate.

Now we show that we can construct a weak solution via the above Galerkin method using a standard compactness argument.

Given g and f and define u_m as above, by above energy estimates we have that $\{u_m\}_m$ is bounded in $L^2([0,T];H^1_0(U))$ and $\{u'_m\}_m$ is bounded in $L^2([0,T];H^{-1}(U))$. By Banach-Alaoglu, we can extract a subsequence u_{m_l} such that u_{m_l} converges weakly to some u in $L^2([0,T];H^1_0(U))$ and u'_{m_l} converges weakly to some v in $L^2([0,T];H^{-1}(U))$. It is a standard check that $v = u'$.

We next want to verify that u is the desired weak solution. As a first step, we want to verify that

$$\langle v, u'\rangle + B[v,u;t] = \langle v, f\rangle \tag{8.5}$$

for v belonging to a dense subset of

$$H := \{v : v \in L^2([0,T];H^1_0(U)),\ v' \in L^2([0,T];H^{-1}(U))\}$$

We may assume v takes form

$$v(t) = \sum_{k=1}^N d^k(t)w_k$$

Now, note that for $m \geqslant N$,

$$\langle u'_m, v\rangle + B[v,u_m;t] = \langle f, v\rangle$$

So integrate in time from 0 to T we get

$$\int_0^T \langle u'_m, v\rangle\, dt + \int_0^T B[v,u_m;t]\, dt = \int_0^T \langle f, v\rangle\, dt$$

for every $m \in \mathbb{N}$. Passing to the subsequence $\{u_{m_l}\}_l$ and sending $l \to \infty$, we get

$$\int_0^T \langle u', v\rangle\, dt + \int_0^T B[v,u;t]\, dt = \int_0^T \langle f, v\rangle\, dt$$

by weak convergence of $\{u_{m_l}\}_l$ and $\{u'_{m_l}\}_l$. Since v is arbitrary in a dense subset of H, we actually have (8.5) as desired. Moreover, we have the lemma below that gives us more regularity on u.

Lemma 8.6. *If $u \in L^2([0,T];H^1_0(U))$ and $u' \in L^2([0,T];H^{-1}(U))$, then actually $u \in C([0,T];L^2(U))$.*

The next step is to verify that $u(0) = g$. Note that for every $v \in H$ with $v(T) = 0$,

$$-\int_0^T \langle u, v' \rangle \, dt + \int_0^T B[v, u; t] \, dt = \int_0^T \langle f, v \rangle \, dt \; + \langle u(0), v(0) \rangle$$

On the other hand, for every u_{m_l} and $v \in H$ with $v(T) = 0$, we have

$$\int_0^T \langle u_{m_l}, v' \rangle \, dt + \int_0^T B[v, u_{m_l}; t] \, dt = \int_0^T \langle f, v \rangle \, dt \; + \langle u_{m_l}(0), v(0) \rangle$$

Sending $l \to \infty$, we know that $u_{m_l}(0) \to g$ in $H_0^1(U)$, so

$$\int_0^T \langle u, v' \rangle \, dt + \int_0^T B[v, u; t] \, dt = \int_0^T \langle f, v \rangle \, dt \; + \langle g, v(0) \rangle$$

Since $v \in H$ is arbitrary, we have $g = u(0)$ as desired.

Eventually we employ energy method to show uniqueness. Note that we only need to show that $u \equiv 0$ when f and g are both identically 0. To this end, we observe that

$$\begin{aligned} \frac{1}{2} \frac{d}{dt} \|u\|_{L^2(U)}^2 &= \langle u', u \rangle = -B[u, u; t] \\ &\leqslant \gamma \|u\|_{L^2(U)}^2 \end{aligned}$$

So by Gronwall and the fact that $u(0) = 0$ we get $\|u(t)\|_{L^2(U)} = 0$ for every $t \in [0, T]$, and thus $u \equiv 0$ as desired.

9. Linear Hyperbolic Equations

We consider the problem

$$u_t + \sum_j B_j u_{x_j} = f, \quad u(0) = g$$

where B_j's are symmetric and C^1. We also assume $g \in H^1$ and f is H^1 in both space and time. We construct the solution by **vanishing viscosity method**. For this, we begin by considering the mollified equation

$$u_t^\varepsilon + \sum_j B_j u_{x_j}^\varepsilon - \varepsilon \Delta u^\varepsilon = f, \quad u(0) = \eta_\varepsilon * g =: g^\varepsilon \tag{9.1}$$

Theorem 9.2. *For each $\varepsilon > 0$, the mollified equation (9.1) has a unique solution $u^\varepsilon \in L_t^2 H_x^3$ with $(u^\varepsilon)' \in L_t^2 H_x^1$*

Theorem 9.3 (Energy Estimates)**.** *Suppose u^ε solves (9.1). Then we have the estimate*

$$\|u^\varepsilon\|_{L^\infty([0,T];H^1)} + \|(u^\varepsilon)'\|_{L^\infty([0,T];L^2)} \leqslant C(\|f\|_{L^\infty([0,T];H^1)} + \|f'\|_{L^\infty([0,T];H^1)} + \|g\|_{H^1}) \tag{9.4}$$

Proof. Standard energy estimates will gives us

$$\frac{d}{dt} \|u^\varepsilon\|_{L^2}^2 \leqslant C(\|u^\varepsilon\|_{L^2}^2 + \|f\|_{L^2}^2)$$

So by Gronwall's inequality,

$$\|u^\varepsilon\|_{L^2}^2 \leqslant C'(\|g\|_{L^2} + \|f\|_{L^2([0,T];L^2)})$$

Now, if we denote $v := u_{x_k}^\varepsilon$, we have

$$v_t + \sum_j B_j v_{x_j} - \varepsilon \Delta v = f - \sum_j (B_j)_{x_k} u_{x_j}$$

so that we can argue again using the standard energy estimates to get

$$\frac{d}{dt}\|v\|_{L^2}^2 \leqslant C(\|\nabla u^\varepsilon\|_{L^2}^2 + \|\nabla f\|_{L^2}^2)$$

and thus

$$\frac{d}{dt}\|\nabla u^\varepsilon\|_{L^2}^2 \leqslant C(\|\nabla u^\varepsilon\|_{L^2}^2 + \|\nabla f\|_{L^2}^2)$$

Using Gronwall as usual, we get

$$\|\nabla u^\varepsilon\|_{L^\infty([0,T];L^2)} \leqslant C(\|\nabla g\|_{L^2}^2 + \|\nabla f\|_{L^2([0,T];L^2)}^2)$$

Eventually, setting $w := u_t$, we have

$$w_t + \sum_j B_j w_{x_j} = f_t - \sum_j (B_j)_t u_{x_j}$$

with initial data

$$w(0) = f(0) - \sum_j B_j g^\varepsilon_{x_j} + \varepsilon \Delta g^\varepsilon$$

Arguing as before, we obtain

$$\|w\|_{L^\infty([0,T];L^2)}^2 \leqslant C(\|f(0)\|_{L^2}^2 + \|\nabla g\|_{L^2}^2 + \varepsilon^2\|\nabla g^\varepsilon\|_{L^2}^2 + \|f_t\|_{L^2([0,T];L^2)})$$

Note that

$$\|f(0)\|_{L^2}^2 \lesssim \|f\|_{L^2([0,T];L^2)} + \|f'\|_{L^2([0,T];L^2)}$$

and that

$$\varepsilon^2\|\nabla g^\varepsilon\|_{L^2}^2 \lesssim \|\nabla g\|_{L^2}^2$$

so combining everything together we get the desired result. $\square$

From the energy estimates, we can prove existence of weak solutions using a compactness argument just as before.

10. Energy Estimates for Water Wave Equations

Here we give an alternative approach to the energy estimates in [8]. Our energy function is

$$E(t) = E_a(t) + E_c(t) + \left\|\partial'_\alpha \frac{1}{Z_{,\alpha'}}\right\|_{L^2}^2 + \|\partial'_\alpha \overline{Z}_t\|_{L^2}^2 + \|D^2_{\alpha'}\overline{Z}_t\|_{L^2}^2 + \left|\frac{1}{Z_{,\alpha'}}(0,t)\right|$$

where

$$E_a(t) = \int \frac{|Z_{,\alpha'} D_t D_{\alpha'}\overline{Z}_t|^2}{A_1} d\alpha' + \|D_{\alpha'}\overline{Z}_t\|_{\dot{H}^{1/2}}$$

$$E_c(t) = \int \frac{|Z_{,\alpha'} D_t^3 D_{\alpha'}\overline{Z}_t|^2}{A_1} d\alpha' + \|D_t^2 D_{\alpha'}\overline{Z}_t\|_{\dot{H}^{1/2}}$$

To close the energy estimates, we mostly imitate [8], and a slightly non-trivial modification is control of $\|D_t^2(\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1})\|_{L^\infty}$. We have the expression

$$\frac{\mathfrak{a}_t}{\mathfrak{a}} \circ h^{-1} = \frac{-\mathrm{Im}(2[Z_t, \mathbb{H}]\overline{Z}_{tt,\alpha'} + 2[Z_{tt}, \mathbb{H}]\overline{Z}_{t,\alpha'} - [Z_t, Z_t; D_{\alpha'}\overline{Z}_t]}{A_1} \tag{10.1}$$

The key is to control the term

$$\|[Z_t, Z_t; D_{\alpha'}\overline{Z}_t]\|_{L^\infty}$$

Note that

$$
\begin{aligned}
|[Z_t, Z_t; D_{\alpha'}\overline{Z}_{ttt}](\alpha')| &= \left| \int \frac{(Z_t(\alpha') - Z_t(\beta'))^2}{(\alpha'-\beta')^2} \frac{1}{Z_{,\alpha'}}(\beta')\partial_{\alpha'}\overline{Z}_{ttt}(\beta')\, d\beta' \right| \\
&\leqslant \int \left| \frac{(Z_t(\alpha') - Z_t(\beta'))^2}{(\alpha'-\beta')^2} \frac{1}{Z_{\alpha'}}(\beta')\partial_{\alpha'}\overline{Z}_{ttt}(\beta') \right| d\beta' \\
&\leqslant \left(\int \left| \frac{Z_t(\alpha') - Z_t(\beta')}{\alpha'-\beta'} \right|^2 d\beta' \right)^{1/2} \\
&\quad \times \left(\int \left| \frac{Z_t(\alpha') - Z_t(\beta')}{\alpha'-\beta'} \frac{1}{Z_{\alpha'}}(\beta')\partial_{\alpha'}\overline{Z}_{ttt}(\beta') \right|^2 d\beta' \right)^{1/2}
\end{aligned}
\tag{10.2}
$$

The first term in the last row is bounded by $\|Z_{t,\alpha'}\|_{L^2}$ by Hardy's inequality. To control the second term, we want to show

$$
\frac{Z_t(\alpha') - Z_t(\beta')}{\alpha'-\beta'} \frac{1}{Z_{\alpha'}}(\beta')
$$

is uniformly bounded for all $\alpha', \beta' \in \mathbb{R}$. Without loss of generality, we may assume $\beta' < \alpha'$, since the other situation is similar. We observe that

$$
\begin{aligned}
\frac{Z_t(\alpha') - Z_t(\beta')}{\alpha'-\beta'} \frac{1}{Z_{,\alpha'}}(\beta') &= \frac{1}{\alpha'-\beta'} \int_{\beta'}^{\alpha'} \frac{1}{Z_{,\alpha'}}(\beta')\; Z_{t,\alpha'}(x)\; dx \\
&= \frac{1}{\alpha'-\beta'} \int_{\beta'}^{\alpha'} \left(\int_x^{\beta'} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}(y) + \frac{1}{Z_{,\alpha'}}(x) \right) Z_{t,\alpha'}(x)\; dydx \\
&= \frac{1}{\alpha'-\beta'} \int_{\beta'}^{\alpha'} D_{\alpha'} Z_t(x)\; dx + \frac{1}{\alpha'-\beta'} \int_{\beta'}^{\alpha'} \int_x^{\beta'} \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}(y) Z_{t,\alpha'}(x)\; dydx
\end{aligned}
$$

Therefore,

$$
\begin{aligned}
\left| \frac{Z_t(\alpha') - Z_t(\beta')}{\alpha'-\beta'} \frac{1}{Z_{,\alpha'}}(\beta') \right| &\leqslant \|D_{\alpha'} Z_t\|_{L^\infty} + \frac{1}{\alpha'-\beta'} \int_{\beta'}^{\alpha'} \int_{\beta'}^{\alpha'} \left| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}}(y) Z_{t,\alpha'}(x) \right| dydx \\
&\leqslant \|D_{\alpha'} Z_t\|_{L^\infty} + \frac{1}{\alpha'-\beta'} (\alpha'-\beta')^{1/2} \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} (\alpha'-\beta')^{1/2} \|Z_{t,\alpha'}\|_{L^2} \\
&\leqslant \|D_{\alpha'} Z_t\|_{L^\infty} + \left\| \partial_{\alpha'} \frac{1}{Z_{,\alpha'}} \right\|_{L^2} \|Z_{t,\alpha'}\|_{L^2}
\end{aligned}
$$

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